Hecke–Kiselman monoids and algebras

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Definition (Ganyushkin, Mazorchuk)

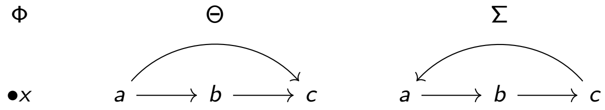
Let Θ be a simple **oriented graph** with n vertices. Then the corresponding Hecke–Kiselman monoid HK_Θ is the monoid generated by **idempotents** $x_1, \ldots, x_n$ such that:

1) if the vertices x_i, x_j are not connected in Θ, then $x_i x_j = x_j x_i$,
2) if x_i, x_j are connected by an arrow $x_i \to x_j$ in Θ, then $x_i x_j x_i = x_j x_i x_j = x_i x_j$.

Hecke–Kiselman algebra $K[\mathrm{HK}_\Theta]$ is the monoid algebra over a field K corresponding to the monoid HK_Θ.

$\rightsquigarrow$ the special case of definition for more general graphs: if the vertices x_i and x_j are connected by an edge $x_i - x_j$, then $x_i x_j x_i = x_j x_i x_j$

Examples



1) Monoid $\mathrm{HK}_\Phi = \langle x \mid x^2 = x \rangle$ consists of two elements $1, x$.
2) Monoid HK_Θ is given by

$$\mathrm{HK}_\Theta = \langle a, b, c \mid a^2 = a, b^2 = b, c^2 = c, \\ ab = aba = bab, bc = bcb = cbc, ac = aca = cac \rangle.$$

It has exactly 18 elements.

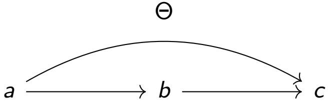
3) The monoid associated to Σ has the following presentation

$$\mathrm{HK}_\Sigma = \langle a, b, c \mid a^2 = a, b^2 = b, c^2 = c, \\ ab = aba = bab, bc = bcb = cbc, ca = aca = cac \rangle.$$

HK_Σ is infinite. Every word in the free monoid $\langle a, b, c \rangle$ can be uniquely rewritten in HK_Σ as a subword of one of the infinite words $cabcab\ldots$ or $cbacba\ldots$.

Motivations

1) Hecke–Kiselman monoids are natural quotients of **Coxeter monoids (**0**–Hecke monoids)**.
2) **Kiselman's semigroups** have origins in the convexity theory. They correspond to the Hecke–Kiselman monoids associated to certain oriented graphs, for example to the graph Θ.



3) If Σ is obtained from the graph Λ by orienting all edges, then HK_Σ is a homomorphic image of the monoid HK_Λ.

$\rightsquigarrow$ Investigation of the Hecke–Kiselman monoids associated to oriented graphs is a natural first step to understand monoids associated to arbitrary graphs.

Properties of Hecke–Kiselman monoids and algebras

Theorem (Mazorchuk)

Monoid HK_Θ is finite $\Longleftrightarrow$ graph Θ is acyclic.

Lemma

Finite Hecke–Kiselman monoids are $\mathcal{J}$-trivial.

Definition

K-algebra R satisfies a polynomial identity (is PI algebra), if there exists a nonzero polynomial $f(x_1, \ldots, x_n)$ in non-commuting variables with coefficients in K, for some $n \geqslant 1$, such that $f(r_1, \ldots, r_n) = 0$ for all $r_1, \ldots, r_n \in R$.

Theorem (Męcel, Okniński)

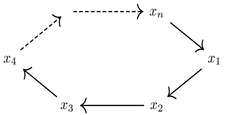
For an oriented graph Θ the following conditions are equivalent

1) $K[\mathrm{HK}_\Theta]$ is PI,
2) graph Θ does not contain two different cycles connected by an oriented path of length $\geqslant 0$,
3) monoid HK_Θ does not contain free submonoid of rank 2.

Monoid C_n and algebra $K[C_n]$ associated to an oriented cycle

Monoid C_n for any $n \geqslant 3$ is given by the presentation

$$\langle x_1, \ldots, x_n : x_i^2 = x_i,\ x_i x_{i+1} = x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1} \text{ for } i = 1, \ldots, n,$$
$$x_i x_j = x_j x_i \text{ for } n-1 > i-j > 1 \rangle$$



What is known about C_n and $K[C_n]$?

1) (Denton) C_n is a $\mathcal{J}$-**trivial monoid**.
2) (Męcel, Okniński) $K[C_n]$ **satisfies a polynomial identity** and is of **Gelfand–Kirillov dimension one**.
3) (Męcel, Okniński) **Gröbner basis** of $K[C_n]$ can be characterized.

$\rightsquigarrow$ (Okniński, MW) description of **reduced forms** of (almost all) elements of C_n

Useful tool: semigroups of matrix type

Definition

If S is a semigroup, A, B are nonempty sets and $P = (p_{ba})$ is a $B \times A$ - matrix with entries in S^0, then the semigroup of matrix type $\mathcal{M}^0(S, A, B; P)$ over S is the set of all matrices of size $A \times B$ with at most one nonzero entry with the operation

$$M \cdot N = M \circ P \circ N$$

for every matrices M and N, where $\circ$ is standard matrix multiplication.

Ideal chain and matrix structures inside C_n

Theorem

C_n has a chain of ideals

$$\emptyset = I_{n-2} \lhd I_{n-3} \lhd \cdots \lhd I_0 \lhd I_{-1} \lhd C_n,$$

with the following properties

1) for $i = 0, \ldots, n-2$ there exist semigroups of matrix type $M_i = \mathcal{M}^0(S_i, A_i, B_i; P_i)$, such that $M_i \subset I_{i-1}/I_i$ (we assume that $I_{n-3}/\emptyset = I_{n-3} \cup \{\theta\}$), where S_i is the infinite cyclic semigroup, P_i is a square symmetric matrix of size $B_i \times A_i$ and with coefficients in $S_i^1 \cup \{\theta\}$;
2) $|A_i| = |B_i| = \binom{n}{i+1}$ for every $i = 0, \ldots, n-2$;
3) for $i = 1, \ldots, n-2$ the sets $(I_{i-1}/I_i) \setminus M_i$ are finite and C_n/I_{-1} is a finite semigroup.

Properties of the monoid C_n and algebra $K[C_n]$

Theorem (MW)

Monoid C_n satisfies a nontrivial semigroup identity.

Definition

Algebra is **right (left) Noetherian**, if every ascending chain of right (left) ideals $I_1 \subset I_2 \subseteq \ldots$ stabilises.

Definition

Algebra A is **semiprime**, if for any ideal $I \triangleleft A$ the condition $I^2 = \{0\}$ implies that $I = \{0\}$.

Theorem (Okniński, MW)

Hecke–Kiselman algebra $K[C_n]$ is a semiprime (right and left) Noetherian algebra.

Corollary

$K[C_n]$ embeds into the matrix algebra $M_r(L)$ over a field L for some $r \geqslant 1$.

Irreducible representations of $K[C_n]$

Let K be an algebraically closed field.

Theorem

Every irreducible representation $\varphi : K[C_n] \longrightarrow M_j(K)$ of the Hecke–Kiselman algebra $K[C_n]$ over an algebraically closed field K is of one of the following forms

1) φ is induced by an irreducible representation of the semigroup of matrix type M_i inside the ideal chain of C_n,
2) φ is 1-dimensional representation associated to an idempotent of the monoid C_n.

$\rightsquigarrow$ characterization of all idempotents of the monoid C_n is known

Irreducible representations of $K[M_i]$

Recall that $M_i = \mathcal{M}^0(S_i, A_i, B_i; P_i)$, where S_i is infinite cyclic semigroup generated by s_i, P_i is a $B_i \times A_i$ matrix with coefficients in $S_i^1 \cup \{\theta\}$.

$$M_i \rightsquigarrow \text{completely 0-simple closure } cl(M_i) = \mathcal{M}^0(\mathrm{gr}(s_i), A_i, B_i; P_i).$$

Theorem

Every irreducible representation of the infinite cyclic group $gr(s_i)$ induces a unique irreducible representation of M_i. It is induced by an irreducible representation of $cl(M_i)$.
Conversely, every irreducible representation of M_i comes from a representation of the group $gr(s_i)$, and can be uniquely extended to an irreducible representation of $cl(M_i)$.

General case: semigroup identities and Noetherian property

Let Θ be an oriented graph.

Theorem (MW)

The following conditions are equivalent

1) the Hecke–Kiselman monoid HK_Θ satisfies a nontrivial semigroup identity,
2) Θ does not contain two different cycles connected by an oriented path of length $\geqslant 0$.

Theorem (Okniński, MW)

The following conditions are equivalent

1) $K[\mathrm{HK}_\Theta]$ is right Noetherian,
2) $K[\mathrm{HK}_\Theta]$ is left Noetherian,
3) every connected component of the graph Θ is either an oriented cycle of some length or an acyclic graph.

General case: the radical of PI Hecke–Kiselman algebras

The Jacobson radical of an algebra A is given by

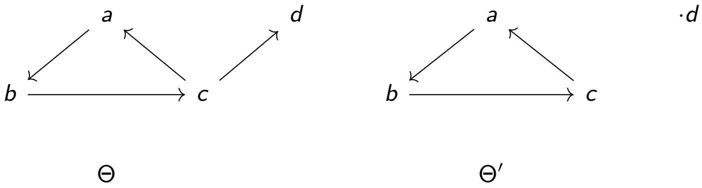
$$\mathcal{J}(A) = \{a \in A \mid aM = 0 \text{ for every simple } A\text{–module } M\}.$$

Let Θ be a graph such that $K[\mathrm{HK}_\Theta]$ satisfies a polynomial identity.

Definition

Denote by Θ' the subgraph of Θ obtained by deleting all arrows $x \to y$ that are not contained in any cyclic subgraph of Θ.

Example

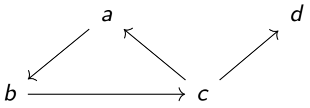


General case: the radical of PI Hecke–Kiselman algebras

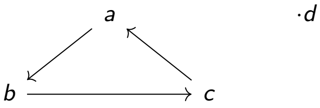
Theorem (Okniński, MW)

1) The Jacobson radical $\mathcal{J}(K[\mathrm{HK}_\Theta])$ of $K[\mathrm{HK}_\Theta]$ is an ideal generated by elements $xy - yx$ for all edges $x \to y$ in Θ that are not contained in any cyclic subgraph of Θ.
2) $K[\mathrm{HK}_\Theta]/\mathcal{J}(K[\mathrm{HK}_\Theta]) \cong K[\mathrm{HK}_{\Theta'}]$, and it is the tensor product of algebras $K[\mathrm{HK}_{\Theta_i}]$ of the connected components $\Theta_1, \ldots, \Theta_m$ of Θ', each being isomorphic to $K \oplus K$ or to the algebra $K[C_j]$, for some $j \geqslant 3$.

Example



Θ



Θ'

$$\mathcal{J}(K[\mathrm{HK}_\Theta]) = \langle dc - cd \rangle, \quad K[\mathrm{HK}_\Theta]/\mathcal{J}(K[\mathrm{HK}_\Theta]) \cong K[\mathrm{HK}_{\Theta'}] \cong K[C_3] \otimes (K \oplus K)$$

General case: irreducible representations of PI Hecke–Kiselman algebras

K is an algebraically closed field.

Let Θ' be the subgraph of Θ obtained by deleting all arrows $x \to y$ that are not contained in any cyclic subgraph of Θ. Denote by $\Theta_1, \dots, \Theta_m$ the connected components of Θ'.

Theorem

Every irreducible representation of $K[\mathrm{HK}_\Theta]$ is of the form

$$K[\mathrm{HK}_\Theta] \to K[\mathrm{HK}_{\Theta_1}] \otimes \cdots \otimes K[\mathrm{HK}_{\Theta_m}] \to M_{r_1}(K) \otimes \cdots \otimes M_{r_m}(K) \xrightarrow{\simeq} M_{r_1 \cdots r_m}(K),$$

where

1) the first map is the natural epimorphism onto $K[\mathrm{HK}_\Theta] \big/ \mathcal{J}(K[\mathrm{HK}_\Theta])$,
2) the second map is the natural map $\psi_1 \otimes \cdots \otimes \psi_m$ for some irreducible representations $\psi_i : K[\mathrm{HK}_{\Theta_i}] \to M_{r_i}(K)$, $i = 1, \dots, m$.

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Thank you!