Can constellations shed light on semigroups?

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Stemming from work with

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Warning: there will be some category theory in this talk.

Even “worse”, there will be generalisations of categories!

But categories are just semigroups with two unary operations (if you add a zero!).

And the generalisations provide nice ways to make semigroups.

1: INTRODUCING CONSTELLATIONS

For any function $f : Y \to Z$ where Y, Z are sets, we have:

- Y is the *domain* of f, $\mathrm{Dom}(f)$;
- Z is the *codomain* of f, $\mathrm{Cod}(f)$.

Composition of functions in the category SET is only defined “rarely”:

for $f : X \to Y$ and $g : Z \to W$, we define

$f \circ g : X \to W$, but *only if* $Y = Z$, or Cod$(f) =$ Dom(g).

(Note: we read compositions left to right, and write “xf”, not “$f(x)$”.)

But for $f \circ g$ to make sense, Wikipedia points out that it is enough to have Cod$(f) \subseteq$ Dom(g).

Actually, if for $f : Y \to Z$, we define

- $\mathrm{Im}(f) = \{xf \mid x \in Y\}$, the *image* of f,

then having $\mathrm{Im}(f) \subseteq \mathrm{Dom}(g)$ is enough!

This gives the *constellation product* $f \cdot g$.

In defining this, codomain ceases to be relevant.

But domain still is.

A category C is a class with a partial binary operation $\circ$ such that, for all $x, y \in C$,

1. $x \circ (y \circ z)$ is defined if and only if $(x \circ y) \circ z$ is defined and then they are equal

2. if $x \circ y, y \circ z$ are defined, so is $x \circ (y \circ z)$

3. for each $x \in C$ there are identities $e, f \in C$ such that $e \circ x = x$, $x \circ f = x$
   (e is an identity if whenever $e \circ s$ or $s \circ e$ exists, it equals s)

The identities making $e \circ x = x$, $x \circ f = x$ are unique and we call then $D(x)$ and $R(x)$ respectively.

Can show the identities of the category are $C^0 = \{D(x) \mid x \in C\}$.

We take a similar approach to defining constellations.

Suppose P is a class with a partial binary operation $\cdot$.

Then $e \in P$ is a *right identity* if it is such that, for all $x \in P$, if $x \cdot e$ is defined then it equals x.

A *(left) constellation* is a class P with a partial binary operation $\cdot$ such that for all $x, y, z \in P$:

(C1) if $x \cdot (y \cdot z)$ exists then so does $(x \cdot y) \cdot z$, and then the two are equal;

(C2) $x \cdot (y \cdot z)$ exists whenever $x \cdot y$ and $y \cdot z$ exist;

(C3) for each $x \in P$, there is a unique right identity $e \in P$ for which $e \cdot x = x$.

We write $D(x) = e$ in (C3) and view P as having signature $(\cdot, D)$.

It follows easily that $D(P) = \{D(x) \mid x \in P\}$ is the set of all right identities in P, and all are idempotent (meaning $e \cdot e = e$).

P is quasiordered via $s \lesssim t \Leftrightarrow s = D(s) \cdot t$ (standard quasiorder).

Note that on $D(P)$, $e \lesssim f$ iff $e \cdot f$ exists.

Every category is a constellation if we ignore R, and then the standard quasiorder is equality.

So constellations generalise categories.

Example: PT_X, the partial functions defined on X,

with constellation product and $D(f) =$ the identity map on Dom(f).

In this example, $s \lesssim t$ iff s is a restriction of t, a partial order.

There is a “Cayley theorem” for small constellations whose standard quasiorder is a partial order (= “normal constellations”):

all embed in PT_X for some X.

Constellations were first defined in [Gould and Hollings, 2009].

They showed that the class of left restriction semigroups corresponds to a special class of small constellations they called *inductive*.

Left restriction semigroups are a variety of unary semigroups with unary operation D of domain (identity function on domain of a function).

Example: PT_X with partial function composition and D!

(The “symmetric left restriction monoid on X”!)

Axioms for left restriction semigroups: semigroups plus

- $D(x)x = x$;

- $D(D(x)) = D(x) = D(x)^2$;

- $D(D(x)y) = D(x)D(y) = D(y)D(x)$;

- $D(xy) = D(xD(y))$;

- $xD(y) = D(xy)x$.

(There is a dual notion of a right restriction semigroup.)

It follows that $D(S) = \{D(s) \mid s \in S\}$ is a semilattice under multiplication,

and that $D(s)$ is the smallest $e \in D(S)$ such that $es = s$.

There is a “Cayley theorem” for left restriction semigroups in terms of PT_X.

To go from a left restriction semigroup S to a constellation,

define $s \cdot t = st$ whenever $sD(t) = s$.

The result is an inductive constellation $(S, \cdot, D)$.

What makes a constellation inductive?

Recall that a constellation has its natural quasiorder:

$s \leq t$ if and only if $D(s) \cdot t$ exists and equals s.

The constellation P is inductive iff

- for all $e \in D(P)$ and $s \in P$ there is a unique largest $x \in P$ such that $x \leq s$ and $x \cdot e$ exists, the *co-restriction* of s to e, denoted $s|e$;

- whenever $x \cdot y$ exists and $e \in P$,
$$D((x \cdot y)|e) = D(x|D(y|e)).$$

This forces $\leq$ to be a partial order,

and $D(P)$ is a semilattice under it with $e \wedge f = e|f$.

The constellation obtained from a left restriction semigroup is inductive.

Conversely, if P is inductive, we may define the *pseudoproduct*

$s \otimes t := s|D(t) \cdot t$ for all $s, t \in P$,

and then $(P, \otimes, D)$ is a left restriction semigroup.

The constructions are mutually inverse, morphisms correspond etc.

Gould and Hollings' work sought to extend earlier work relating semigroups having both domain and range notions to certain types of ordered categories:

see the ESN-theorem, as well as [Lawson, 1991].

Already then, (certain special types of) constellations connect directly to semigroup theory.

[Lawson, 1991]
M.V. Lawson, 'Semigroups and ordered categories I: the reduced case'. *J. Algebra* 141 (1991), 422–462.

[Gould and Hollings, 2009]
V. Gould and C. Hollings, 'Restriction semigroups and inductive constellations'. *Comm. Algebra* 38 (2009), 261–287.

2: RIGHT CANONICAL EXTENSION: CATEGORIES FROM CONSTELLATIONS

We define the constellation $CSET$ using constellation product of *surjective* functions:

with $f \cdot g$ defined if and only if $\mathrm{Im}(f) \subseteq \mathrm{Dom}(g)$,

and with D defined as in the category SET.

This gives a composable constellation

(for every s there exists t such that $s \cdot t$ exists).

The concrete categories of groups GRP, topological spaces TOP, posets etc also have corresponding constellations:

restrict to surjective morphisms and use the constellation product and D.

Thus we get $CGRP$, $CTOP$ and so on.

These examples of constellations have an obvious notion of range:

(the identity map on) the image of a morphism.

Let P be a constellation.

Let $\mathcal{C}(P) = \{(s, e) \in P \times D(P) \mid s \cdot e = s\}$.

On $\mathcal{C}(P)$, define $(s, e) \circ (t, f) = (s \cdot t, f)$

providing $e = D(t)$ (in which case $s \cdot t$ exists).

Also define $D((s, e)) = (D(s), D(s))$ and $R((s, e)) = (e, e)$.

Proposition: For any constellation P, $(\mathcal{C}(P), \circ, D, R)$ is a category.

We call $(\mathcal{C}(P), \circ, D, R)$ here the *canonical extension* of the constellation P.

If P is a category, then it is easy to see that $P \cong \mathcal{C}(P)$.

Generally, P is a quotient of $\mathcal{C}(P)$ in a certain sense.

A similar construction was considered by Lawson in [Lawson, 2004].

He showed how to turn an ordered groupoid into a left cancellative category.

But ordered groupoids can be viewed as constellations (with inverses),

and then his construction becomes a special case of this one.

We're interested in the congruences θ on a category K such that

K/θ is a constellation and $K \cong \mathcal{C}(K/\theta)$.

Call them *canonical congruences*: they have a nice internal description.

It's possible to define these on general (composable) constellations.

There is a Correspondence Theorem for them,

so maximal $\leftrightarrow$ simple, etc.

We can define a canonical congruence δ on SET as follows:

$(f, g) \in \delta$ if and only if Dom$(f) =$ Dom(g), Im$(f) =$ Im(g),

and $xf = xg$ for all $x \in$ Dom(f).

Moreover $SET/\delta \cong CSET$, and so $SET \cong \mathcal{C}(CSET)$.

And $CSET$ is “canonically simple” (so irreducible).

So this is a “best possible” representation of SET as $\mathcal{C}(P)$.

A similar argument can be given for the category of groups:

$GRP \cong \mathcal{C}(CGRP)$, although $CGRP$ is *not* canonically simple.

But every composable constellation has a canonically simple quotient (by Zorn).

This means there is some canonically simple quotient P of $CGRP$,

and then $GRP \cong \mathcal{C}(P)$ is “best possible”.

(How about rings, topological spaces, posets etc?)

So constellations seem pretty useful in the study of categories.

[Gould and S., 2017] : V. Gould and T. Stokes, ‘Constellations and their relationship with categories’. *Algebra Universalis* 77 (2017), 271–304.

See also:

[Gould and S., 2022]

V. Gould and T. Stokes, ‘Constellations with range and IS-categories’.

J. Pure Appl. Algebra 226 (2022), 106995.

How about for semigroups?

After all, constellations were invented to give an ESN-type theorem for left restriction semigroups!

3: LEFT CANONICAL EXTENSIONS.

There is a way to obtain constellations themselves via canonical extension of even more primitive objects, that “adds in” the domains of elements!

These primitive objects are partial algebras with distinguished ‘idempotents” that can be used to extend on the left.

We can apply the construction to semigroups to obtain constellations;

sometimes these constellations are inductive, hence are “really” just left restriction semigroups!

Basic idea: we have a semigroup S with some idempotents $E \subseteq S$;

we define $\mathcal{C}_E(S) = \{(e, s) \in E \times S \mid es = s\}$,

and we set $(e, s) \cdot (f, t) = (e, st) \in C_E(S)$, whenever $sf = s$,

and define $D((e, s)) = (e, e) \in \mathcal{C}_E(S)$.

Then $(\mathcal{C}_E(S), \cdot, D)$ is always a constellation!

Sometimes this constellation $\mathcal{C}_E(S)$ is inductive.

This means that the constellation product on $C_E(S)$ can be completed to a left restriction semigroup operation in a unique way.

Indeed, for any semigroup S with $E \subseteq E(S)$,

one can determine exactly when $\mathcal{C}_E(S)$ is inductive.

It turns out there must be an action of S on E satisfying certain conditions.

And E must be a meet-semilattice wrt a relevant quasiorder.

Specifically:

- the quasiorder on E given by $e\omega^l f$ if $e = ef$ is a partial order and E is a meet-semilattice under it;
- $s \cdot (e \wedge f) = (s \cdot e) \wedge (s \cdot f)$, where $\wedge$ is meet in E;
- for all $s, t \in S$ and $e \in E$, $ste = st$ iff $s(t \cdot e) = s$.

Call a pair (S, E) satisfying these necessary and sufficient conditions an inductive left E-monoid.

Then S does indeed act on E in the usual sense: $(st) \cdot e = s \cdot (t \cdot e)$.

Denote by $Rest(E, S)$ the left restriction semigroup you get by extending the product on $C_E(S)$ in this way when (S, E) is an inductive left E-monoid.

For any $(e, s), (f, t) \in Rest(E, S)$, their product is

$$(e, s) \otimes (f, t) = (e \wedge (s \cdot f), (e \wedge (s \cdot f))st),$$

where $e \wedge f$ is the meet of $e, f \in E$ and $s \cdot f$ is the action of s on f.

$Rest(E, S)$ restricts back to give $C_E(S)$ if we keep D and limit products: $s \cdot t := st$ providing $sD(t) = s$.

Which left restriction semigroups R arise from inductive constellations $C_E(S)$? Let's answer this for monoids.

Exactly ones with "enough large idempotents" in the following sense.

For every $e \in D(R)$, there is $e' \in E(R)$ with $D(e') = 1$ and $e \mathcal{L} e'$.

Now let $R_1 = \{s \in R \mid D(s) = 1\}$ and $E = \{e' \mid e \in D(R)\}$.

Then (R_1, E) is an inductive left E-monoid, in which $s \cdot e' = D(se)'$ for all $s \in R_1$ and $e' \in E$,

and $R \cong Rest(E, S)$! (The exact choice of E doesn't matter!)

So...which left restriction monoids have enough large idempotents?

The archetypal left restriction semigroup is PT_X...

and it has enough large idempotents!

Here, $(PT_X)_1 = T_X$, and for $e \in D(PT_X)$, let $e' \in T_X$ be any projection onto $dom(e)$.

Then define $E = \{e' \mid e \in D(PT_X)\}$.

So (T_X, E) is an inductive left E-monoid, and $PT_X \cong Rest(E, T_X)$.

But the general theory still works fine with this modification.

So let's work through an example, with $X = \{x, y\}$.

Then $PT_X = \{1, 0, p_x, p_y, e, f, i, j, k\}$, where

1 is the identity map, 0 is the empty function,

$p_x = \{(x, x), (y, x)\}$, $p_y = \{(x, y), (y, y)\}$,

$e_x = \{(x, x)\}$, $e_y = \{(y, y)\}$,

$i = \{(x, y), (y, x)\}$, $j = \{(x, y)\}$, $k = \{(y, x)\}$.

Then $D(PT_X) = \{1, 0, e_x, e_y\}$,

$E(PT_X) = D(PT_X) \cup \{p_x, p_y\}$, and

$(PT_X)_1 = T_X = \{1, p_x, p_y, i\}$ and $E((PT_X)_1) = \{1, p_x, p_y\}$.

The idempotents in the $\mathcal{L}$-classes are $\{0\}, \{1\}, \{p_x, e_x\}$ and $\{p_y, e_y\}$.

So $1 \ \mathcal{L} \ 1$, $e_x \ \mathcal{L} \ p_x$, $e_y \ \mathcal{L} \ p_y$,

witnessing the fact that PT_X has almost enough large idempotents.

So we let $E = \{0, 1, p_x, p_y\}$ and build $C^0_E(S)$ equal to

$$\{(0,0), (p_x, p_x), (p_y, p_y), (p_x, p_y), (p_y, p_x), (1,1), (1,i), (1,p_x), (1,p_y)\},$$

corresponding to partial functions as follows:

$$(0,0) \leftrightarrow \emptyset,\ (p_x, p_x) \leftrightarrow e_x,\ (p_y, p_y) \leftrightarrow e_y,\ (p_x, p_y) \leftrightarrow j,$$

$$(p_y, p_x) \leftrightarrow k,\ (1,1) \leftrightarrow 1,\ (1,i) \leftrightarrow i,\ (1,p_x) \leftrightarrow p_x,\ (1,p_y) \leftrightarrow p_y.$$

For example, since $D((p_x, p_y)) = (p_x, p_x)$, and $(p_x, p_y) = (p_x, p_x)(1, p_y)$,

we interpret (p_x, p_y) as “the restriction of p_y to Ran(p_x)”, or e_y.

So $C_E^0(T_X^0)$ correctly enumerates the elements of PT_X.

And constellation products are trivial to compute:

e.g. recall that $i = \{(x, y), (y, x)\}$ and $j = \{(x, y)\}$.

So $ji = j \cdot i = \{(x, x)\} = e_x$.

Correspondingly, we have $(p_x, p_y) \cdot (1, i) = (p_x, p_y i) = (p_x, p_x)$, For general compositions, we use multiplication in $Rest_0(E, T_X^0)$.

For example, to get $ij = e_y$ in PT_X, we proceed to compute

$$(1, i) \otimes (p_x, p_y) = (1 \wedge (i \cdot p_x), (1 \wedge (i \cdot p_x))ip_y),$$

which requires calculation of $i \cdot p_x = p_y$ (the largest left equalizer in E of 1 and p_x). Hence,

$$(1, i) \otimes (p_x, p_y) = (1 \wedge (i \cdot p_x), (1 \wedge (i \cdot p_x))ip_y)$$

$$= (p_y, p_y ip_y) = (p_y, p_x p_y) = (p_y, p_y),$$

as it should!

New kind of example: consider the set Rel_X of binary relations on X

and equip it with domain D defined as for PT_X.

Then $(Rel_X, \cdot, D)$ is not a left restriction monoid.

BUT instead equip Rel_X with demonic composition $*$:

$(x, y) \in s * t$ iff $(x, y) \in st$ AND $xs \subseteq dom(t)$.

Like usual relational composition, this operation is associative...

and it generalises both partial function composition and composition of left total binary relations.

Indeed $(Rel_X, *, D)$ is known to be a left restriction monoid.

Here, $(Rel_X)_1$ is the semigroup of left total binary relations Rel^t_X, whose domains are all of X.

And $(Rel_X, *, D)$ has almost enough large idempotents:

for $e \in D(Rel_X)$, let e' be as for PT_X, so define E as there.

Then $C^0_E(Rel^t_X)$ is inductive, and $Rest^0(E, Rel^t_X) \cong (Rel_X, *, D)$.

A further example comes from the partition monoid $P(X)$ on the set X.

Consider the submonoid $P^t(X)$ of all “left total” partitions $P^t(X)$ on X (defined in the “obvious” way).

It’s not hard to see this is a *right* restriction monoid where $R(s)$ is the finest block partition e such that $se = s$.

And it’s got enough large idempotents!

What is $P^t(X)_1$?

Basically a copy of T_X!

And the large idempotents consist of one transformation for each equivalence relation on X.

So we must have $P^t(X) \cong RRest((T_X, E)$!

(The "dual symmetric left restriction monoid on X"!)

[S., 2022] : T. Stokes, 'Left restriction monoids from left E-completions'. *J. Algebra* 608 (2022), 143–185.

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