Duality theory for right restriction semigroups

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Ehresmann semigroups

A left Ehresmann semigroup is an algebra $(S; \cdot, ^+)$ of type $(2, 1)$, where $(S; \cdot)$ is a semigroup and the following identities hold:

$$x^+x = x, \ \ x^+y^+ = y^+x^+ = (x^+y^+)^+, \ \ (xy)^+ = (xy^+)^+.$$

Right Ehresmann semigroups are defined dually by the identities:

$$xx^* = x, \ \ x^*y^* = y^*x^* = (x^*y^*)^*, \ \ (xy)^* = (x^*y)^*.$$

A two-sided Ehresmann semigroup, or an Ehresmann semigroup, is an algebra $(S; \cdot, ^*, ^+)$ which combines left and right Ehresmann semigroups so that

$$(x^+)^* = x^+, \ \ (x^*)^+ = x^*.$$

The identities $x^+ = x^+x^+$ and $(x^+)^+ = x^+$ and their dual identities $x^* = x^*x^*$ and $(x^*)^* = x^*$ follow from the axioms.

The semilattice of projections

Let S be an Ehresmann semigroup. The set

$$P(S) = \{s^*\colon s \in S\} = \{s^+\colon s \in S\}$$

is a semilattice and $e^* = e^+ = e$ holds for all $e \in P(S)$. It is called the semilattice of projections of S and its elements are called projections.

The following identities can be easily derived from the definition:

$$\forall s \in S, e \in P(S)\colon\ (se)^* = s^*e,\ \ (es)^+ = es^+.$$

The natural partial orders

Let S be an Ehresmann semigroup. For $a, b \in S$, we put:

- $a \leq_l b$ if there is $e \in P(S)$ such that $a = eb$;
- $a \leq_r b$ if there is $e \in P(S)$ such that $a = be$;
- $a \leq b$ if there are $e, f \in P(S)$ such that $a = ebf$.

$\leq_l$, $\leq_r$ and $\leq$ are partial orders and are called the natural left partial order, the natural right partial order and the natural partial order.

$a \leq_l b$ holds if and only if $a = a^+b$ and, dually, $a \leq_r b$ holds if and only if $a = ba^*$.

Restricted to $P(S)$, all the orders coincide.

The monoid of binary relations

Let X be a non-empty set and let $\mathcal{B}(X)$ be the monoid of binary relations on X with the operation of the composition of relations. By $\mathrm{id}_X = \{(x, x) : x \in X\}$ we denote the identity relation.
$\mathcal{B}(X)$ is an Ehresmann monoid if one defines τ^+ and τ^* by

$$\tau^+ = \mathrm{ran}(\tau) = \{(x, x) \in X \times X : \exists y \in X \text{ such that } (x, y) \in \tau\},$$

$$\tau^* = \mathrm{dom}(\tau) = \{(y, y) \in X \times X : \exists x \in X \text{ such that } (x, y) \in \tau\}.$$

Projections of $\mathcal{B}(X)$ are:

$$P(\mathcal{B}(X)) = \{\tau \in \mathcal{B}(X) : \tau \subseteq id_X\}$$

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Restriction and birestriction semigroups

A left restriction semigroup is an algebra $(S;\cdot,^+)$ which is a left Ehresmann semigroup and satisfies the identity

$$xy^+ = (xy)^+x.$$

A right restriction semigroup dually satisfies the identity

$$x^*y = y(xy)^*$$

A two-sided restriction semigroup or a birestriction semigroup or just a restriction semigroup is both left and right restriction semigroup.
The next identities are called the left ample identity and the right ample identity:

$$\forall s \in S, e \in P(S)\colon \; se = (se)^+s, \;\; es = s(es)^*.$$

Restriction semigroups generalize inverse semigroups. If S is an inverse semigroup and $a \in S$, we define $a^+ = aa^{-1}$ and $a^* = a^{-1}a$.

Compatibility and bicompatibility

Let S be a right restriction semigroup and $a, b \in S$. Define $a \sim b$ if $ab^* = ba^*$. If the join $a \vee b$ exists with respect to $\leq_r$ then $a \sim b$. We call such join a compatible join.

Let S be a two-sided restriction semigroup. Define $a \approx b$ if $ab^* = ba^*$ and $b^+a = a^+b$. We say that a and b are bicompatible. If the join $a \vee b$ with respect to $\leq$ exists then $a \approx b$.

If S is an inverse semigroup then $a \approx b$ if and only if $a \sim b$ and $a^{-1} \sim b^{-1}$.

Nota bene. Bicompatibility can not be defined in a right restriction semigroup. We do need either a two-sided restriction semigroup or an inverse semigroup (which induces the two-sided restriction structure).

Basic examples

- The partial transformation monoid $\mathcal{PT}(X)$ is a $(2,1,1,0)$-subalgebra of $\mathcal{B}(X)$ and is a right restriction Ehresmann monoid.
- The symmetric inverse monoid $\mathcal{I}(X)$.
- $\mathcal{UI}(X) = \{f \in \mathcal{I}(X) \colon f(x) \leq x \text{ for all } x \in \mathrm{dom}(f)\}$ - upper triangular rook matrices. It is a two-sided restriction monoid, it is ample since it satisfies:

$$ab = ac \Longrightarrow ab^+ = ac^+ \quad \text{and} \quad ba = ca \Longrightarrow b^*a = c^*a.$$

Cayley and ESN-type representations

	ENS-type	Cayley-type
inverse semigroups	Inductive groupoids	Yes
ample semigroups	Inductive categories	Yes
right restriction semigroups	Inductive constellations	Yes
right restriction Ehresmann	right Ehresmann categories	Yes[1]
regular $*$-semigroups	Chained projection groupoids	?

[1] provided that $ab = ac \implies ab^+ = ac^+$ holds, Schein (1970)

Many object generalizations

Inverse monoid	$\to$	Inverse category
Right restriction monoid	$\to$	Restriction category
Right Ehresmann monoid	$\to$	Support category
Two-sided Ehresmann monoid	$\to$	Bisupport category
Right restriction Ehresmann monoid	$\to$	Range category
Projection	$\to$	Restriction idempotent

Observation. (Cockett, Guo, Hofstra, 2012) Let C be a support category. If it admits a compatible cosupport then the latter is unique.

Corollary (for one object). Let $(S; \cdot, ^*)$ be a right Ehresmann monoid. If it admits a unary operation $^+$ making $(S; \cdot, ^*, ^+)$ a (two-sided) Ehresmann monoid then such an operation $^+$ is unique.

When is a right restriction monoid Ehresmann?

Let S be a right restriction monoid. An element $s \in S$ is called open, if the map $\varphi_s \colon P(S) \to P(S)$, $\varphi_s(e) = (es)^*$, has a left adjoint $\psi_s \colon P(S) \to P(S)$ and the Frobenius condition holds, for all $e, e' \in P(S)$:

$$\psi_s(e\varphi_s(e')) = \psi_s(e)e'.$$

Proposition (Cockett, Guo, Hofstra, 2012) (proved for range categories) Let S be a right restriction semigroup. The following are equivalent:

1. S admits a unary operation $^+$ such that $(S; \cdot, ^*, ^+)$ is a right restriction Ehresmann semigroup.
2. All elements of S are open.

If the conditions hold, one defines $s^+ = \psi_s(1)$ for all $s \in S$.

Partial isomorphisms

Let S be a right restriction semigroup. An element $s \in S$ is called a partial isomorphism if there is $s' \in S$ (partial inverse) such that $s's = s^*$ and $ss' = (s')^*$ (Jackson and Stokes, 2009, true inverses; Cockett, Guo, Hofstra, 2012).

Proposition

1. *If s is a partial isomorphism then its partial inverse is unique and is a partial isomorphism.*
2. *If S is an inverse semigroup then every element s is a partial isomorphism which is just s^{-1}.*
3. *If every element is a partial isomorphism, then S is an inverse semigroup and $s' = s^{-1}$ for all s.*
4. *The set of partial isomorphisms $\mathrm{Inv}(S)$ is closed with respect to the multiplication and is thus an inverse semigroup with $s^{-1} = s'$.*

Bideterministic elements

Let S be an Ehresmann semigroup. An element $a \in S$ is right deterministic if for all $e \in P(S)$: $ea = a(ea)^*$. It is left deterministic if for all $e \in P(S)$: $ae = (ae)^+a$. It is bideterministic if it is both left and right deterministic. Notation: $\mathcal{D}(S)$ - the set of bideterministic elements.

Proposition

1. $\mathcal{D}(S)$ is a $(2,1,1)$-subalgebra of S and is a restriction semigroup.
2. $\mathrm{Inv}(S) \subseteq \mathcal{D}(S)$.
3. If S is right restriction Ehresmann, then bideterministic elements coincide with left deterministic ones.

Example. This inclusion is strict, for example, for the monoid $\mathcal{UI}_n$ of upper-triangular rook $n \times n$ matrices.

Classical Stone duality

By GBA we denote the category of generalized Boolean algebras and proper morphisms between them, and by LCBS the category of locally compact Boolean spaces and proper continuous maps between them.

- Let E be a GBA. By $\widehat{E}_p$ we denote the set of all prime characters of E, basis of the topology: $D_e = \{f \in \widehat{E}_p \colon f(e) = 1\}$, $e \in E$.
- Let X be a LCBS. By $GBA(X)$ we denote the GBA of all compact-open sets of X.

Theorem (Stone duality for generalized Boolean algebras, Stone 1937, Doctor 1964) The assignments $E \mapsto \widehat{E}_p$ and $X \mapsto GBA(X)$ give rise to contravariant functors $\mathcal{F} \colon \mathrm{GBA} \to \mathrm{LCBS}$ and $\mathrm{G} \colon \mathrm{LCBS} \to \mathrm{GBA}$ which establish a dual equivalence between the categories GBA and LCBS.

Dualities for Boolean inverse and restriction semigroups

An inverse semigroup S is Boolean if $E(S)$ is a generalized Boolean algebra and joins of bicompatible pairs of elements exist in S.
A topological groupoid is ample if it is étale and its space of identities is a locally compact Boolean space.

Theorem 1. The category of Boolean inverse semigroups is dually equivalent to the category of ample groupoids (Lawson, 2010, 2012, Lawson and Lenz, 2013, GK and Lawson, 2017, richer functoriality).

Theorem 2. Abstract complete pseudogroups are dual to localic étale groupoids (Resende, 2007, functoriality 2015). This extends the duality between frames and locales (Johnstone, Stone spaces).

Theorem 3. (GK and Lawson, 2017)

- Complete restriction monoids are dual to localic étale categories.
- Boolean restriction monoids are dual to topological ample categories.

Duality for join restriction categories (Cockett and Garner)

Cockett and Garner (2021): let $\mathcal{C}$ be a join restriction category with local glueings. Duality between:

- join restriction categories with a well-behaved functor to $\mathcal{C}$ (hyperconnected over $\mathcal{C}$),
- partite source-étale internal categories in $\mathcal{C}$.

For the 'one object case' this specializes to a duality between:

- complete pseudomonoids $\theta\colon S \to \mathcal{PT}(X)$, X is an object of $\mathcal{C}$, that is, S is a join restriction monoid and θ is a morphism of restriction monoids which induces an isomorphism $P(S) \to P(\mathcal{PT}(X))$.
- source-étale internal categories in $\mathcal{C}$.

Local glueings

A local atlas on an object X of a join restriction category is a symmetric matrix of projections (projection idempotents) $\varphi_{ij} \in P(X)$, $i, j \in I$, such that

$$\varphi_{ij}\varphi_{jk} \leq \varphi_{ik}.$$

If $p = \vee_i p_i$ is a local homeomorphism $A \to X$ and $s_i \colon X \to A$ its partial inverse of p_i, the family $\varphi_{ij} = p_j s_j$ is a local atlas on X.

A join restriction category $\mathcal{C}$ has local glueings if every local atlas φ on every object X is induced by some local homeomorphism $p \colon A \to X$.

Examples. The category Top_p of topological spaces (and partial maps) has local glueings - the germ construction; the category Set_p of sets, the category Loc_p of locales, among others.

Join inverse categories

A join inverse category is an inverse category with joins of bicompatible families $\{s_i \colon i \in I\}$.

A join restriction category is étale (Cocket and Garner, 2021) if every element is a (right) join of partial isomorphisms.

Proposition. The categories of join inverse categories and étale join restriction categories are equivalent.

In particular, the category of Boolean inverse monoids is equivalent to the category of étale Boolean right restriction monoids.

Enlightening example. $\mathcal{I}(X)$ – Boolean inverse monoid, assign to it its 'right join completion' — $\mathcal{PT}(X)$, this is an étale Boolean right restriction monoid. Reverse direction: assign to $\mathcal{PT}(X)$ the Boolean inverse monoid of partial isomorphisms — $\mathcal{I}(X)$.

How powerful is the Cockett-Garner duality theory?

The Kudryavtseva-Lawson duality for Boolean (two-sided) restriction semigroups does not follow from the Cockett-Garner duality theory.

Example. Let $\mathcal{UPT}_n = \{f \in \mathcal{PT}_n \colon f(x) \leq x \text{ for all } x \in \mathrm{dom}(f)\}$ — upper-triangular $0-1$ matrices, in each column at most one 1. This is right restriction monoid, there is a dual right ample category: objects $\{1, 2, \ldots, n\}$, maps (x, y), $x \leq y$.

$\mathcal{UPT}_n$ is poor in partial isomorphisms. By taking partial isomorphisms, we get just projections. But what we really want to capture is $\mathcal{UI}_n$ – the partial bijections inside $\mathcal{UPT}_n$. For this we need the notion of bideterministic elements and thus cosupport. This is not covered in Cocket and Garner, 2021.

Our contribution briefly

- replace monoids by semigroups,
- more structure: not just right restriction semigroups, but right restriction Ehresmann semigroups,
- argue that bideterministic elements are an appropriate generalization of partial isomorphisms,
- generalize the notion of an étale right restriction monoid: a right restriction Ehresmann semigroup is étale, if every element is a join of bideterministic elements.

Remark. In the groupoidal case, bideterministic elements coincide with partial isomorphisms, and the range operation is not needed to be mentioned explicitly as it is determined by the domain operation and the inversion.

Boolean right restriction semigroups

Definition

Let S be a right restriction semigroup with a left zero 0, which is a projection. It is called Boolean if the following conditions are satisfied.

1. For any two elements $a, b \in S$ such that $a \sim b$, the join $a \vee b$ exists in S.
2. $(P(S), \leq)$ is a generalized Boolean algebra.
3. For any $a, b, c \in S$ where $a \sim b$ we have $(a \vee b)c = ac \vee bc$.

We say that a Boolean right restriction semigroup is supported if it satisfies the next condition.

(S) For every $a \in S$ there is $e \in P(S)$ such that $ea = a$.

A Boolean right restriction Ehresmann semigroup is called étale (GK, 2024) if every element is a (right) join of bideterministic elements.

Right ample categories

A topological category

$$C = (C_1, C_0, u, d, r, m)$$

is an internal category in the category of topological spaces and suppose that C_0 is a locally compact Boolean space. We call a topological category C

- right ample if $d \colon C_1 \to C_0$ is a local homeomorphism,
- ample if both $d \colon C_1 \to C_0$ and $r \colon C_1 \to C_0$ are local homeomorphisms.

Proposition. Suppose $C = (C_1, C_0, u, d, r, m)$ is right ample. Then u is open and m is open (Tristan Bice).

The map r being continuous, is not in general open or a local homeomorphism (in contrast to ample groupoids).

The Boolean right restriction semigroup $\mathsf{S}(C)$

Let $C = (C_1, C_0, u, d, r, m)$ be a right ample category. A right slice is an open subset A of C_1 such that the restriction of d to A is injective. Put $\mathsf{S}(C)$ to be the set of all compact right slices. Define the operations: $A^* = ud(A)$ and the multiplication is induced by the category multiplication.

Proposition. (GK, 2024)

- $\mathsf{S}(C)$ is a supported Boolean right restriction semigroup.
- If r is open, $\mathsf{S}(C)$ admits a structure of a right restriction Ehresmann semigroup with $A^+ = ur(A)$.
- If r is a local homeomorphism $\mathsf{S}(C)$ is étale.

If r is étale, let $\widetilde{\mathsf{S}}(C)$ be the Boolean restriction semigroup of two-sided slices.

The category of germs $\mathsf{C}(S)$ of a supported Boolean right restriction semigroup S

Let C_0 be the space of prime characters of $P(S)$. For $e \in P(S)$ we put $D_e = \{\varphi \in X : \varphi(e) = 1\}$. For $\varphi \in D_{s^*}$ define the map $s \circ \varphi \in C_0$ by

$$s \circ \varphi \colon P(S) \to \{0, 1\}, \ (s \circ \varphi)(e) = \varphi((es)^*).$$

We put $\Omega = \{(s, \varphi) \in S \times X : \varphi \in D_{s^*}\}$. For $(s, \varphi), (t, \psi) \in \Omega$ put $(s, \varphi) \sim (t, \psi)$ iff $\varphi = \psi$ and $\exists u \leq s, t$ with $\varphi \in D_{u^*}$. Let $C_1 = \Omega / \sim$ be the set of germs. Define the maps $d, r \colon C_1 \to C_0$ and $u \colon C_0 \to C_1$ by

$$d([s, \varphi]) = \varphi, \ r([s, \varphi]) = s \circ \varphi, \ u(\varphi) = [e, \varphi],$$

where $e \in P(S)$ is such that $\varphi(e) = 1$.
For $([s, \varphi], [t, \psi]) \in C_2$ define the map $m \colon C_2 \to C_1$,

$$[s, \varphi] \cdot [t, \psi] = [st, \psi].$$

Then $\mathsf{C}(S) = (C_1, C_0, d, r, u, m)$ is a right ample category.

Properties of C(S)

Let $\Theta(s)$ to be the set of all the germs $[s, \varphi] \in C_1$ – all germs over s. The sets $\Theta(s)$ are compact right slices and form the base of the topology on C(S). One can check that $\Theta(st) = \Theta(s)\Theta(t)$, $d(\Theta(s)) = D_{s^*}$, for all $s, t \in S$.

Proposition (GK, 2024)

- Suppose S is right restriction Ehresmann. Then r is open and $r(\Theta(s)) = D_{s^+}$.
- Suppose S is right restriction Ehresmann. An element $s \in S$ is bideterministic iff $\Theta(s)$ is a two-sided slice.
- Suppose S is étale. Then r is a local homeomorphism.
- Suppose S is étale. Then the category attached to $\mathcal{D}(S)$ coincides with C(S).

The dualitites - 1

Theorem A. (GK, 2024)

1. The category of Boolean right restriction semigroups is dually equivalent to the category of right ample categories.
2. The category of Boolean right restriction Ehresmann semigroups is dually equivalent to the category of right ample categories whose range map r is open.
3. The category of étale Boolean right restriction Ehresmann semigroups is dually equivalent to the category of ample categories.

Morphisms

- Between semigroups: proper and weakly meet-preserving $(2,1)$-homomorphisms $f\colon S \to T$ so that $f\colon P(S) \to P(T)$ is a morphism of GBAs.
- Between categories: proper and continuous right covering functors

Generalization is possible, to cofunctors (Cockett, Garner, 2021), same idea étale space cohomomorphisms (GK, 2012).

The dualitites - 2

Theorem B. (GK, 2024) The category of étale Boolean right restriction Ehresmann semigroups and morphisms that preserve deterministic elements is dually equivalent to the category of ample categories and proper and continuous covering functors. (This is obtained by restricting morphisms in part 3 of Theorem A.)

Corollary 1. (GK, Lawson, 2017) The category of Boolean two-sided restriction semigroups is dually equivalent to the category of ample categories.

Corollary 2. (GK, 2024) The category of étale Boolean right restriction semigroups is equivalent to the category of Boolean two-sided restriction semigroups.

Groupoidal étale right restriction semigroups

Call an étale Boolean right restriction semigroup S groupoidal if its dual category is a groupoid. This is equivalent to requiring that $\mathcal{D}(S) = \mathrm{Inv}(S)$ and we recover the definition of étale by Cocket and Garner. Restricting objects in Corollary 2 (and restricting to monoids), we recover the following.

Corollary 3. (Cockett and Garner, 2021) The category of groupoidal étale Boolean right restriction monoids is equivalent to the category of Boolean inverse monoids semigroups.

Generalizations?

We anticipate that the following generalizations are possible:

- Boolean right restriction Ehresmann monoids $\rightarrow$ join range categories.
- Categories $\rightarrow$ semicategories.
- Generalizations to objects over any join range cagegory $\mathcal{C}$ with local glueings.

More results and future work

Let S be a right restriction semigroup.

More results.

- We can assign to S the universal category and the tight category. These are right ample.
- The K-algebra of a right restriction semigroup is isomorphic to the K-algebra of its universal category (generalizing Steinberg - groupoid approach to discrete inverse semigroup algebras, 2010).

Future work.

- Study these algebras, connect their properties with the properties of the category.
- Find interesting examples.
- Connect this with the existing literature on non self-adjoint subalgebras of groupoid C^*-algebras.