# Prefix monoids of groups and right units of special inverse monoids


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# The "driving engine" (part I)

H. H. Wilhelm Magnus (1930/31):

*The word problem for every one-relator group* $\mathrm{Gp}\langle A \mid r = 1\rangle$ *is decidable.*

Reason (the Magnus-Moldavansky hierarchy):

- $G = \mathrm{Gp}\langle A \mid r = 1\rangle$ embeds into an HNN-extension of its (f.g.) subgroup $L = \mathrm{Gp}\langle A' \mid r' = 1\rangle$ w.r.t. a pair of free ("Magnus") subgroups of $L$, where $|r'| < |r|$;
- This suffices to reduce the WP for $G$ to that of $L$;
- Eventually, we end up with a free group of finite rank, where we trivially solve the WP.
- NB. There is an older approach (Magnus' original) using amalgamated free products.

# The "driving engine" (part II)

Open problem (as of 20 June 2024):

*Does every one-relator monoid*
$\mathrm{Mon}\langle A \mid u = v\rangle$ *have a decidable WP?*

S.I.Adian (1966) – The word problem for $\mathrm{Mon}\langle A \mid u = v\rangle$ is decidable for:

- *special monoids – the def. relation is of the form $u = 1$,*
- *the case when both $u, v$ are non-empty, and have different initial letters and different terminal letters.*

Adian & Oganessian (1987) – The general problem reduces to two particular cases:

- $\mathrm{Mon}\langle a, b \mid aUb = aVa\rangle$,
- $\mathrm{Mon}\langle a, b \mid aUb = a\rangle$ (the "monadic" case).

NB. These presentations define right cancellative monoids.

# The Lead Role #1: Prefix monoids (in groups)

Let $G = \mathrm{Gp}\langle A \mid w_i = 1 \ (i \in I)\rangle$ be a group.

The prefix monoid of this group (presentation) = the submonoid of $G$ generated by the elements represented by all prefixes of all $w_i$'s

The prefix monoid is dependent on the concrete presentation of $G$ – one fixed (isomorphism type of a) group can have many presentations, leading to many prefix monoids.

Prefix Membership Problem (PMP): Given a word over $A \cup A^{-1}$, decide whether it represents an element of the prefix monoid (w.r.t. the given group presentation)

IgD & RDG (TrAMS, 2021): A kaleidoscope of sufficient conditions (via amalgamated products and HNN extensions) ensuring decidability for the PMP

# The Lead Role #2: Right units (in inverse monoids)

Let $M$ be an inverse monoid.

$r \in M$ is a right unit $\iff r \mathscr{R} 1 \iff rr^{-1} = 1$

Fun facts:

- Right units of $M$ form a right cancellative submonoid $R$ of $M$.
- If $M = \mathrm{Inv}\langle A \mid w_i = 1 \ (i \in I)\rangle$ (i.e. $M$ is a special inverse monoid) then $R$ is generated by elements represented by all prefixes of all $w_i$'s.
- So, in the natural map $M \to G = \mathrm{Gp}\langle A \mid w_i = 1 \ (i \in I)\rangle$, the RU-monoid $R$ of $M$ is mapped onto the prefix monoid of $G$.
- If $M$ happens to be $E$-unitary, the restriction of this map to $R$ is a monoid isomorphism.
- Consequently, the RU-monoid of any $E$-unitary special inverse monoid (SIM) is group-embeddable.

# The "driving engine" (part III)

Ivanov, Margolis & Meakin (JPAA, 2001):
The (right cancellative) monoid $\mathrm{Mon}\langle A \,|\, aUb = aVc\rangle$ $(b \neq c)$ embeds (as the monoid of right units) into

$$\mathrm{Inv}\langle A \,|\, aUbc^{-1}V^{-1}a^{-1} = 1\rangle.$$

Similarly, $\mathrm{Mon}\langle A \,|\, aUb = a\rangle$ embeds into $\mathrm{Inv}\langle A \,|\, aUba^{-1} = 1\rangle$.

Hence, the WP for one-relator monoids reduces to the WP for one-relator inverse monoids.

Fun facts: when $w$ is cyclically reduced then

- $\mathrm{Inv}\langle A \,|\, w = 1\rangle$ is $E$-unitary;
- the WP for $\mathrm{Inv}\langle A \,|\, w = 1\rangle$ reduces to the PMP for $\mathrm{Gp}\langle A \,|\, w = 1\rangle$.

# Surprise, surprise...!

RDG (Inventiones, 2020):

*There exists a one-relator special inverse monoid with an undecidable WP.* [!!!]

Fun facts:

- the counterexample(s) is/are even $E$-unitary;
- at the heart of the proof is Lohrey-Steinberg's result (JAlg, 2008) that the RAAG $A(P_4)$ has a fixed f.g. submonoid with undecidable membership;
- then, $A(P_4)$ embeds into a one-relator group $G = \mathrm{Gp}\langle a, b \mid \ldots \rangle$;
- finally, a one-relator SIM $M = \mathrm{Inv}\langle a, b, t \mid \ldots \rangle$ is constructed so that $u \in \{a, b, a^{-1}, b^{-1}\}^*$ represents an element of the "critical" undecidable f.g. submonoid of $G \iff tut^{-1}$ is a right unit in $M$.

Still, this does not invalidate the IMM aprroach.

# Know your limits

Guba (1997):
For any monadic $M = \mathrm{Mon}\langle a, b \,|\, aUb = a\rangle$ constructs $G_M = \mathrm{Gp}\langle x, y, A \,|\, xWyx^{-1} = 1\rangle$ (for some $W \in (A \cup \{x, y\})^*$) such that the WP for $M$ reduces to PMP for $G_M$.

However, there are groups $G = \mathrm{Gp}\langle A \,|\, w = 1\rangle$ with:

- $w$ reduced and undecidable PMP for $G$ (IgD, RDG, 2021);
- $w = uv^{-1}$ reduced ($u, v \in A^+$) and undecidable PMP for $G$ (Foniqi, RDG, CFNB, to appear);
- $w \in A^+$ and undecidable submonoid membership problem for $G$ (again, FGNB).

# Mon vs Inv

Obviously (imagine Snape's voice here), one-relator/f.p. special monoids and special inverse monoids are very different creatures. For example:

- the group of units $U$ of a $M = \mathrm{Mon}\langle A \mid w = 1\rangle$ is a one-relator/f.p. group;
- the RU-monoid of $M$ is a free product of $U$ and a free monoid of finite rank;
- all other maximal subgroups of $M$ are $\cong U$.

In contrast:

- the group of units $U$ of a $M = \mathrm{Inv}\langle A \mid w = 1\rangle$ can be non-one-relator (RGD, Ruškuc, Jussieu, to appear);
- the RU-monoid of $M$ can be even non-f.p.;
- other maximal subgroups of $M$ can be wildly different from $U$.

# The questions

All of this very much justifies the study of prefix monoids in f.p. groups and RU-monoids in f.p. SIMs in their own right.

(1) What can the prefix monoids of f.p. groups be?

(2) What can the RU-monoids of f.p. SIMs be?

(3) What are the possible groups of units of these monoids?

(4) What are the possible Schützenberger groups of these monoids?

# Recursive stuff

A group $G$ is recursively presented if

$$G = \mathrm{Gp}\langle A \mid w_i = 1\ (i \in I)\rangle$$

where $A$ is finite and $\{w_i : i \in I\}$ is a r.e. language over $A \cup A^{-1}$.

Similarly, a monoid is recursively presented if

$$M = \mathrm{Mon}\langle A \mid u_i = v_i\ (i \in I)\rangle$$

where $A$ is finite and $\{(u_i, v_i) : i \in I\}$ is a r.e. subset of $A^* \times A^*$.

The Higman Embedding Theorem: A finitely generated group embeds into a f.p. group if and only if it is recursively presented.

☞ A finitely generated monoid embeds into a f.p. group if and only if it is group-embeddable and recursively presented.

☞ Every prefix monoid (of a f.p. group) is f.g.
$\Longrightarrow$ it is recursively presented.

# The characterisation of prefix monoids (of f.p. groups)

Two (easy) facts:

- Every group-embeddable f.p. monoid arises as a prefix monoid.
- If a group arises as a prefix monoid then it is f.p. So, not all group-embeddable recursively presented monoids are prefix monoids.

Theorem (IgD, RDG, 2023):
*For every group-embeddable recursively presented monoid $M$ there is a natural number $\mu_M$ such that*

$$M * \Sigma_k^*$$

*is a prefix monoid (with $|\Sigma_k| = k$) if and only if $k \geq \mu_M$.*

Also:
*The class of groups of units of prefix monoids is precisely the recursively presented groups.*

# Recursively enumerable stuff

Let $G$ be a f.p. group (generated by $A$). Let $L \subseteq (A \cup A^{-1})^*$ be a recursively enumerable language such that the set of all elements of $G$ represented by words from $L$ forms a subgroup $H \leq G$. Then $H$ is said to be a recursively enumerable subgroup of $G$.

NB. A r.e. subgroup of $G$ is not necessarily finitely generated. However, all f.g. (i.e. recursively presented) subgroups of $G$ are r.e.

Theorem (IgD, RDG, 2023):
*A group $H$ arises as a Schützenberger group of a prefix monoid (of a f.p. group) $\Longleftrightarrow$ $H$ arises as a r.e. subgroup of a f.p. group.*

Ingredients:

- $M$ (left/right) cancellative $\Longrightarrow$ every Sch-group embeds into the group of units of $M$.
- For every r.e. subgroup $H$ of a f.p. group $G$ there is a f.p. overgroup $G_1 \geq G$ and and $t \in G_1$ such that $G \cap t^{-1}Gt = H$.

# RU-monoids (take 1)

Again, some (easy) facts:

- Every RU-monoid is a right cancellative recursively presented monoid.
- If the monoid of right units of a f.p. SIM is a group $\Longrightarrow$ it is f.p.

Theorem 1 (RDG, Kambites, JEMS, to appear):
*The class of groups of units of f.p. SIMs (and thus of RU-monoids) is precisely the recursively presented groups.*

Theorem 2 (RDG, Kambites):
*A group arises as a maximal subgroup (i.e. as a group $\mathscr{H}$-class) of a f.p. SIM $\Longleftrightarrow$ it arises as a r.e. subgroup of a f.p. group.*

# RC-presentations

$$M = \mathsf{MonRC}\langle A \mid \mathfrak{R} \rangle$$

$\Leftrightarrow M \cong A^*/\mathfrak{R}^{\mathrm{RC}}$, where $\mathfrak{R}^{\mathrm{RC}}$ is the intersection of all congruences $\sigma$ of $A^*$ such that

- $\mathfrak{R} \subseteq \sigma$,
- $A^*/\sigma$ is right cancellative.

A.J.Cain (2005) (+ Robertson, Ruškuc, 2008): A concept of formal, syntactic derivation for RC-presentations.

Theorem (IgD, RDG, 2023):
*Every finitely RC-presented monoid is an RU-monoid.*

In a way, this is a generalisation of the Ivanov-Margolis-Meakin result.

# RU-monoids (take 2)

Theorem (IgD, RDG, 2023):
*The class of Schützenberger groups of RU-monoids is exactly the class of r.e. subgroups of f.p. groups.*

Open Problem: Characterise the class of all RU-monoids.

In the remainder of the talk, I'll present two interesting phenomena in this vein discovered by IgD+RDG during this Spring's online sessions.

# The RU-monoid in the Gray-Ruškuc construction (1)

RDG, Ruškuc: For any group $G$ (f.p. or not) and f.g. submonoid $T$ of $G$, a(n $E$-unitary) SIM $M$ is constructed (which is f.p. when $G$ is) such that:

- $U(M) \cong G * U(T)$,
- if the monoid of right units of $M$ is f.p. so must be both $G$ and $T$.

With the right choice of parameters, this produces:

- a one-relator SIM whose group of units is not one-relator;
- a one-relator SIM whose group of units is f.p. but whose RU-monoid is not f.p.;
- a f.p. SIM whose group of units is not f.p.

# The RU-monoid in the Gray-Ruškuc construction (2)

IgD, RDG (2024):
The RU-monoid of $M$ = the greatest right cancellative image of the HNN-like Otto-Pride extension of $G$ w.r.t. $T \hookrightarrow G$ =

$$\mathrm{MonRC}\langle A, B, t \mid u_i = v_i \ (i \in I),\ tw_j = b_j t \ (j \in J)\rangle$$

where $G = \mathrm{Mon}\langle A \mid u_i = v_i \ (i \in I)\rangle$ and $T = \langle w_j :\ j \in J\rangle_G$.

Hence:

- If $G$ is f.p. then the RU-monoid of $M$ is necessarily finitely RC-presented;
- The group of units $U(M)$ can still be not f.p., and also the RU-monoid can be not f.p. (as a monoid!);
- There is a finitely RC-presented monoid $S$ in which the complement of the group of units $S \setminus U$ is an ideal, and still $U$ is not f.p.

Conclusion: RC-presentations are strange animals!

# The Gray-Kambites construction (1)

Realising an arbitrary recursively presented group as the group of units of a f.p. SIM. Here we present a slight generalisation (by IgD & RDG).

$T = \mathrm{MonRC}\langle A \mid u_i = v_i \ (i = 1, \dots, k)\rangle$
$S = \langle B \rangle_T$ – a f.g. submonoid

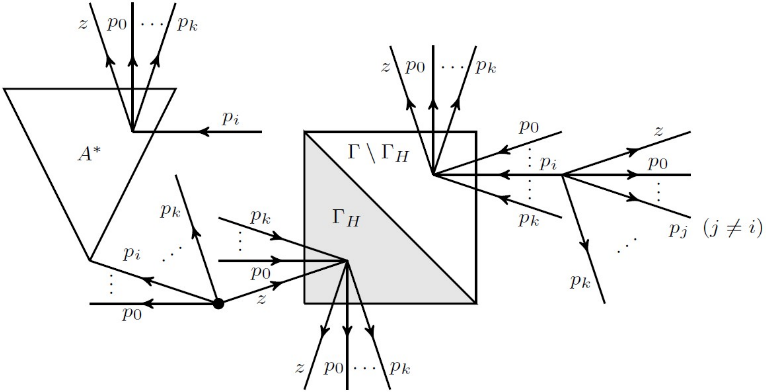
$M_{T,S}$ – a f.p. SIM gen. by $A$ and $p_0, p_1, \dots, p_k, z, d$ subject to

$$p_i a p_i^{-1} p_i a^{-1} p_i^{-1} = 1 \qquad (a \in A,\ i = 0, 1, \dots, k)$$
$$p_i u_i d^{-1} v_i^{-1} p_i^{-1} = 1 \qquad (i = 1, \dots, k)$$
$$p_0 d p_0^{-1} = 1$$
$$z b z^{-1} z b^{-1} z^{-1} = 1 \qquad (b \in B)$$
$$z \left( \prod_{i=0}^{k} p_i^{-1} p_i \right) z^{-1} = 1.$$

# The Gray-Kambites construction (2)

RDG, Kambites (JEMS, to appear): When $T = G$ (a group given by a finite special monoid pres.) and $S = H$ (a f.g. subgroup), then

$$U(M_{G,H}) \cong H.$$



# The Gray-Kambites construction (3)

So, what is the RU-monoid of $M_{T,S}$?

IgD, RDG (2024): RC-presented by $p_i$, $q_i$ $(= zp_i^{-1})$ $(0 \leq i \leq k)$, $a^{(i)}$ $(= p_i a p_i^{-1})$ $(a \in A,\ 0 \leq i \leq k)$, $b^{(z)}$ $(= zbz^{-1})$ $(b \in B)$, and relations

$$\begin{aligned}
&q_i w^{(i)} p_i = q_0 w^{(0)} p_0 && (w \in A^*,\ i = 1, \dots, k)\\
&q_i u^{(i)} = q_i v^{(i)} && (u, v \in A^* \text{ s.t. } u = v \text{ holds in } T,\\
& && i = 0, 1, \dots, k)\\
&q_i b^{(i)} = b^{(z)} q_i && (b \in B,\ i = 0, 1, \dots, k)
\end{aligned}$$

NB. For all $u, v \in B^*$ s.t. $u = v$ holds in $S$, $u^{(z)} = v^{(z)}$ can be RC-derived. In fact, $\langle b^{(z)} :\ b \in B \rangle \cong S$.

# The Gray-Kambites construction (4)

For example, when we take $T = \{a\}^*$ and $S = \langle\varnothing\rangle = \{1\}$ (and a silly presentation for $T$, say $a = a$, to have $k = 1$) we get the RU-monoid

$$\mathrm{MonRC}\langle a_1, a_1, p_0, p_1, q_0, q_1 \mid q_0 a_0^n p_0 = q_1 a_1^n p_1 \ (n \geq 0)\rangle.$$

This can be shown to be:

- not finitely RC-presented,
- with a trivial group of units.

Conclusion: There are non-finitely RC-presented RU-monoids out there!

# Thank you! 😊 ❤️