Congruences on direct products of simple semigroups

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Warning for Vicky

A will be a semigroup or monoid throughout.

Definitions

Definition

Let S and A be semigroups. The *direct product* of S and A is the set

$$S \times A = \{(s, a) \mid s \in S,\ a \in A\}$$

with multiplication $(s, a)(t, b) = (st, ab)$

Definition

A congruence ρ on S is an equivalence relation such that

$$(s, t), (u, v) \in \rho \implies (su, tv) \in \rho$$

- In particular, $\rho \subseteq S \times S$ is a subsemigroup

Definition

A semigroup is simple if it has no proper ideals.

An objective

- We want to describe congruences on $S \times A$ using congruences on S and congruences on A.
- Some congruences are easy, if π_S is a congruence on S and π_A is a congruence on A then

$$\pi_S \times \pi_A = \{((s,a),(t,b)) \mid (s,t) \in \pi_S,\ (a,b) \in \pi_A\}$$

  is a congruence on $S \times A$ - these are called *decomposable congruences*.
- Congruences are subsemigroups, a congruence on $S \times A$ is a subsemigroup of $(S \times A) \times (S \times A)$.
- $(S \times A) \times (S \times A) \cong S^2 \times A^2$ so we could equivalently ask for subsemigroups of $S^2 \times A^2$.

Fibre products

Definition: Fibre product

If S and A are semigroups and there are onto homomorphisms

$$S \xrightarrow{f} V \xleftarrow{g} A$$

then

$$\{(s,a) \in S \times A \mid (s)f = (a)g\}$$

is a subsemigroup of $S \times A$. Called a *fibre product*.

- In general, **not** all subsemigroups of $S \times A$ are fibre products
- Direct products are fibre products
- For groups, all subgroups are fibre products (Goursat's Lemma)

Fibre congruences

We can construct fibre products of congruences on semigroups. Take π_S and π_A congruences on S and A with

$$\pi_S \xrightarrow{f} V \xleftarrow{g} \pi_A$$

Then

$$\{((s_1, a_1), (s_2, a_2)) \mid (s_1, s_2) \in \pi_S,\ (a_1, a_2) \in \pi_A,\ (s_1, s_2)f = (a_1, a_2)g\}$$

is a subsemigroup of $(S \times A) \times (S \times A)$.

Lemma

If S and A are simple monoids then this relation is a congruence on $S \times A$ if and only if V is an abelian group with

- identity $(s, s)f = (a, a)g$ (all $s \in S$, $a \in A$)
- $((s_1, s_2)f)^{-1} = (s_2, s_1)f$, $((a_1, a_2)g)^{-1} = (a_2, a_1)g$

Call congruences of this form *fibre congruences*

Fibre congruences

So we have a collection of congruences on $S \times A$

- Contains all congruences of the form $\pi_S \times \pi_A$
- So contains the identity and universal congruences

Is this all congruences?

- **No**... Rees congruences on direct products are not necessarily fibre
- But sometimes actually... **Yes**

Theorem

If S and A are simple monoids then every congruence on $S \times A$ is a fibre congruence.

Group images

Two ingredients for fibre congruence:

- congruences on factors
- group homomorphic images of congruences (with extra properties)

Theorem (Gomes)

For a semigroup X, group homomorphic images are determined by the *normal subsemigroups*.

- So, equivalently we want to find suitable normal subsemigroups $X \trianglelefteq \pi_S$

Theorem

Let S be a simple monoid and let π be a congruence on S. The normal subsemigroups of π which define suitable group images are precisely the congruences on S which are normal subsemigroups of π.

Congruences on simple monoids

Theorem

Let S, A be simple monoids and let

- π_S be a congruence on S - $\{(s_1, s_2) \mid \exists a_1, a_2 \in A\ (s_1, a_1)\ \rho\ (s_2, a_2)\}$
- π_A be a congruence on A - $\{(a_1, a_2) \mid \exists s_1, s_2 \in S\ (s_1, a_1)\ \rho\ (s_2, a_2)\}$
- $\kappa_S \trianglelefteq \pi_S$ be a congruence on S - $\{(s_1, s_2) \mid (s_1, 1)\ \rho\ (s_2, 1)\}$
- $\kappa_A \trianglelefteq \pi_A$ be a congruence on A - $\{(a_1, a_2) \mid (1, a_1)\ \rho\ (1, a_2)\}$
- $f : \pi_S/\kappa_S \to \pi_A/\kappa_A$ be an isomorphism

Then

$$\{((s_1, a_1), (s_2, a_2)) \mid (s_1, s_2) \in \pi_S,\ (a_1, a_2) \in \pi_A,\ [(s_1, s_2)]f = [(a_1, a_2)]\}$$

is a congruence on $S \times A$. Moreover, all congruences on $S \times A$ are of this form.

Example: Bicyclic monoid

$$B = \langle b, c \mid bc = 1 \rangle$$

- The congruences on B are Δ, $\kappa = \{(c^i b^j, c^m b^n) \colon i - j = m - n\}$ and $\pi_d = \{(c^i b^j, c^m b^n) \colon d \mid (i - j) - (m - n)\}$ for $d \in \mathbb{N}$
- Equivalently, the homomorphic images of B are: B, $\mathbb{Z} \cong B/\kappa$ or $\mathbb{Z}f$ for a group homomorphism f.
- Want to know when congruences are normal subsemigroups of each other.
- If $\pi_1 \trianglelefteq \pi_2 \neq \Delta$ then $\kappa \subseteq \pi_1$, and $\kappa \trianglelefteq \pi_d$
- $\pi_d \trianglelefteq \pi_c$ if and only if $c \mid d$

Theorem

The homomorphic images of $B \times B$ are (up to isomorphism):

- $B \times B$
- $B \times G$ for G an image of $\mathbb{Z}$
- an image of $\mathbb{Z} \times \mathbb{Z}$.

Decomposable congruences

Question from way back at the start:
π_S is a congruence on S and π_A is a congruence on A then

$$\pi_S \times \pi_A = \{((s,a),(t,b)) \mid (s,t) \in \rho_S,\ (a,b) \in \rho_A\}$$

is a congruence on $S \times A$ (a *decomposable* congruence)

Question:
When are all the congruences on $S \times A$ decomposable?

This is known for groups (Miller 1975)

Decomposable congruences

Suppose S and A are non trivial monoids

- If S is not simple and $I \subset S$ be an ideal then $I \times A$ is an ideal of $S \times A$. The Rees congruence, $\rho_{I \times A}$, defined by this ideal is not decomposable.
- In this case we know all the congruences on $S \times A$, the fibre congruences.

The fibre congruence defined by

$$\pi_S \xrightarrow{f} V \xleftarrow{g} \pi_A$$

is decomposable if and only if V is trivial.

Decomposable congruences

So every congruence is decomposable if and only if there are no common abelian group images of a congruence on S and a congruence on A.

Theorem

If S and A are monoids then every congruence on $S \times A$ is decomposable if and only if

- S and A are simple
- for each $\pi_S \in \text{Cong}(S)$ and each $\pi_A \in \text{Cong}(A)$, the orders of the elements of the abelian group homomorphic images (satisfying the homomorphism conditions) of π_S and π_A are relatively prime (in particular are finite).

Height of direct products

Definition

The *height* of a semigroup, $\mathrm{Ht}(S)$, is the maximum length of a chain of congruences.

- If J is a $\mathcal{J}$-class of a semigroup S then the principal factor is $J^* = J \cup \{0\}$.
- J^* is a 0-simple (S, S)-biact.
- Heights can be computed using the heights of the principal factors, regarded as (S, S)-biacts.

Theorem (B, East, Miller, Mitchell, Ruškuc)

If S has n $\mathcal{J}$-classes $J_1, \dots J_n$ then

$$\mathrm{Ht}(S) = \sum_{i=1}^{n} \mathrm{Ht}(J_i^*) - n$$

Heights of direct products

If S and A are monoids then the $\mathcal{J}$-classes of S times A are $J \times K$ where J is a $\mathcal{J}$-class of S and K is a $\mathcal{J}$-class of A
It is possible to construct fibre congruences on products of acts.

Lemma

If J is a $\mathcal{J}$-class of S and K is a $\mathcal{J}$-class of T then every $S \times T$-act congruence on the principal factor $(J \times K)^*$ is a fibre congruence.
Moreover,

$$\mathrm{Ht}((J \times K)^*) = \mathrm{Ht}(J^*) + \mathrm{Ht}(K^*) - 1$$

Theorem

Let S be a monoid with n $\mathcal{J}$-classes and let A be monoid with m $\mathcal{J}$-classes. Then

$$\mathrm{Ht}(S \times A) = m\,\mathrm{Ht}(S) + n\,\mathrm{Ht}(A).$$

Thanks for listening