# The meet-stalactic and meet-taiga monoids

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# Tableaux and Trees

## Stalactic monoids

lSt, rSt

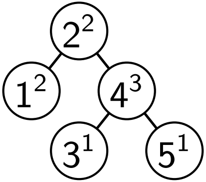
Stalactic tableaux

| | | | | |
|---|---|---|---|---|
| 2 | 1 | 3 | 5 | 4 |
| 2 | 1 | | 5 | |
| | 1 | | | |

## Taiga monoids

lTg, rTg

Binary search trees with multiplicities



# The left and right stalactic and taiga monoids

Let lSt and rSt denote the left and right stalactic monoids, and lTg and rTg denote the left and right taiga monoids.

## Definition

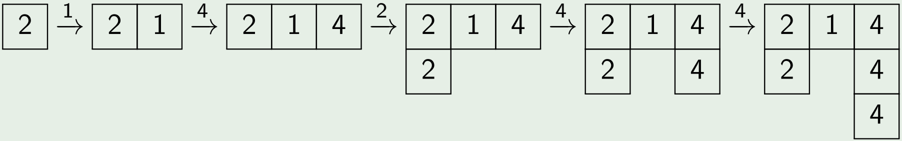
For $u, v \in \mathbb{N}^*$,

$$\begin{aligned} u \equiv_{\mathsf{lSt}} v &\iff P_{\mathsf{lSt}}(u) = P_{\mathsf{lSt}}(v), \\ u \equiv_{\mathsf{rSt}} v &\iff P_{\mathsf{rSt}}(u) = P_{\mathsf{rSt}}(v), \\ u \equiv_{\mathsf{lTg}} v &\iff P_{\mathsf{lTg}}(u) = P_{\mathsf{lTg}}(v), \text{ and} \\ u \equiv_{\mathsf{rTg}} v &\iff P_{\mathsf{rTg}}(u) = P_{\mathsf{rTg}}(v). \end{aligned}$$

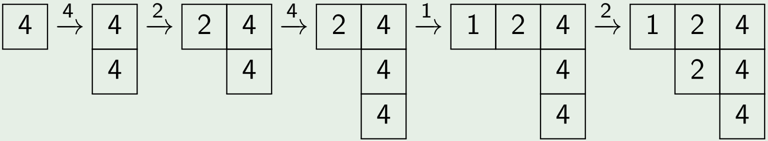
# The stalactic monoid

Given $w = 214244 \in \mathbb{N}^*$, we calculate $P_{\mathrm{lSt}}(w)$ and $P_{\mathrm{rSt}}(w)$.

## Example ($P_{\mathrm{lSt}}(214244)$)



## Example ($P_{\mathrm{rSt}}(214244)$)



# Patience sorting tableau

## Definition

We define increasing [decreasing] patience sorting tableaux by example. Let,

$A =$

| | | |
|---|---|---|
| 1 | 3 | 4 |
| 2 | | 5 |
| 6 | | |

, and $B =$

| | | |
|---|---|---|
| 2 | 5 | 6 |
| | 1 | 4 |
| | | 3 |

.

1. $A$ and $B$ increase left-to-right in the first row.
2. $A$ increases top-to-bottom in each column, and
3. $B$ decreases top-to-bottom in each column.

So $A$ is an *increasing* patience sorting tableau and $B$ is a *decreasing* patience sorting tableau.

# $Q$-symbols for stalactic

Let $w = 214244$.

## Example

$$P_{\mathrm{lSt}}(w) = \begin{array}{|c|c|c|} \hline 2 & 1 & 4 \\ \hline 2 & & 4 \\ \cline{1-1}\cline{3-3} & & 4 \\ \cline{3-3} \end{array} \text{ and } Q_{\mathrm{lSt}}(w) = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & 5 \\ \cline{1-1}\cline{3-3} & & 6 \\ \cline{3-3} \end{array}.$$

## Example

$$P_{\mathrm{rSt}}(w) = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline & 2 & 4 \\ \cline{2-3} & & 4 \\ \cline{3-3} \end{array} \text{ and } Q_{\mathrm{rSt}}(w) = \begin{array}{|c|c|c|} \hline 2 & 4 & 6 \\ \hline & 1 & 5 \\ \cline{2-3} & & 3 \\ \cline{3-3} \end{array}.$$

# Robinson-Schensted-like

**Theorem**

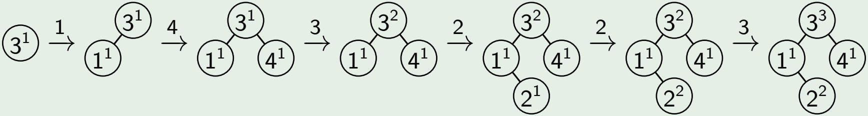
*The map $w \mapsto (P_{\mathsf{lSt}}(w), Q_{\mathsf{lSt}}(w))$ [$w \mapsto (P_{\mathsf{rSt}}(w), Q_{\mathsf{rSt}}(w))$] is a bijection between the elements of $\mathbb{N}^*$ and the set formed by the pairs $(T, S)$ where*

1. *$T$ is a stalactic tableau;*
2. *$S$ is an increasing [decreasing] patience-sorting tableau;*
3. *$T$ and $S$ have the same shape.*

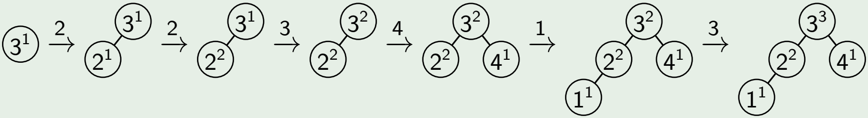
# The taiga monoid

Given $w = 3143223 \in \mathbb{N}^*$, we calculate $P_{\mathrm{lTg}}(w)$ and $P_{\mathrm{rTg}}(w)$.

## Example ($P_{\mathrm{lTg}}(3143223)$)



## Example ($P_{\mathrm{rTg}}(3143223)$)

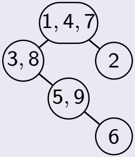


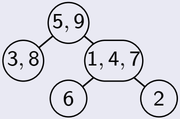
We call, $P_{\mathrm{lTg}}(w)$ and $P_{\mathrm{rTg}}(w)$, *Binary search trees with multiplicities* (BSTM).

# Binary trees over sets

## Definition

We define an *increasing [decreasing] binary tree over sets* (BTS) by example. Let,

$A =$ 

, and $B =$ 

.

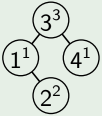
1. $A$ and $B$ contain exactly the interval $\{1, \ldots, m\}$ for some $m \in \mathbb{N}$,
2. replacing each node in $A$ with its minimum obtains an increasing tree,
3. replacing each node in $B$ with its maximum obtains a decreasing tree.

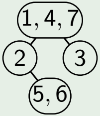
So $A$ is an increasing BTS and $B$ is an decreasing BTS.

# $Q$-symbols for stalactic

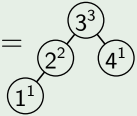
Let $w = 3143223$.

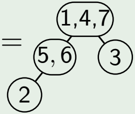
### Example

$P_{\mathrm{lTg}}(w) =$ 

and $Q_{\mathrm{lTg}}(w) =$ 


### Example

$P_{\mathrm{rTg}}(w) =$ 

and $Q_{\mathrm{rTg}}(w) =$ 

# Robinson-Schensted-like

## Theorem

*The map* $w \mapsto (P_{\mathrm{lTg}}(w), Q_{\mathrm{lTg}}(w))$ *[*$w \mapsto (P_{\mathrm{rTg}}(w), Q_{\mathrm{rTg}}(w))$*] a bijection between the elements of* $\mathbb{N}^*$ *and the set formed by the pairs* $(T, S)$ *where*

1. $T$ *is a BSTM;*
2. $S$ *is an increasing [decrasing] BTS such that the union of the sets labelling* $S$ *is the interval* $[m]$*, where* $m$ *is the sum of the multiplicities of* $T$*;*
3. $T$ *and* $S$ *have the same underlying binary tree shape;*
4. *the multiplicity of the* $i$*-th node of* $T$ *is the cardinality of the set labelling the* $i$*-th node of* $S$*.*

# The meet-stalactic and meet-taiga monoid

Let mSt and mTg denote the meet-stalactic and meet taiga monoid.

**Definition**

Let $P_{\mathrm{mSt}}(w) = (P_{\mathrm{lSt}}(w), P_{\mathrm{rSt}}(w))$, and $P_{\mathrm{mTg}}(w) = (P_{\mathrm{lTg}}(w), P_{\mathrm{rTg}}(w))$. Then, for $u, v \in \mathbb{N}^*$,

$$u \equiv_{\mathrm{mSt}} v \iff P_{\mathrm{mSt}}(u) = P_{\mathrm{mSt}}(v), \text{ and}$$
$$u \equiv_{\mathrm{mTg}} v \iff P_{\mathrm{mTg}}(u) = P_{\mathrm{mTg}}(v).$$

# Example

We can equally define a $Q$-symbol for mSt by
$Q_{\mathrm{mSt}}(w) = (Q_{\mathrm{lSt}}(w), Q_{\mathrm{rSt}}(w))$

## Example

$$P_{\mathrm{mSt}}(214244) = \left( \begin{array}{|c|c|c|} \hline 2 & 1 & 4 \\ \hline 2 & & 4 \\ \cline{1-1}\cline{3-3} & & 4 \\ \cline{3-3} \end{array}, \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline & 2 & 4 \\ \cline{2-3} & & 4 \\ \cline{3-3} \end{array} \right) \text{ and}$$

$$Q_{\mathrm{mSt}}(214244) = \left( \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & 5 \\ \cline{1-1}\cline{3-3} & & 6 \\ \cline{3-3} \end{array}, \begin{array}{|c|c|c|} \hline 2 & 4 & 6 \\ \hline & 1 & 5 \\ \cline{2-3} & & 3 \\ \cline{3-3} \end{array} \right)$$

# Twin stalactic tableaux

Let $\mathrm{cont}(T)$ be the multi-set containing every label in $T$. We say a column is *simple* if it only contains one block.

**Definition**

Let $T_L, T_R$ be stalactic tableaux. We say $(T_L, T_R)$ are a *pair of twin stalactic tableaux* if

1. $\mathrm{cont}(T_L) = \mathrm{cont}(T_R)$, and
2. for each simple column labelled $c$ and any column labelled $d$, if $c$ is left of $d$ in $T_L$, then $c$ is left of $d$ in $T_R$.

**Proposition**

*For any $w \in \mathbb{N}^*$, the meet-stalactic P-symbol $(P_{\mathrm{lSt}}(w), P_{\mathrm{rSt}}(w))$ of $w$ is a pair of twin stalactic tableaux.*

# Twin patience-sorting tableaux

## Definition

Let $(S_L, S_R)$ be a pair of (respectively) increasing and decreasing patience-sorting tableaux. We say $(S_L, S_R)$ are a *pair of twin patience-sorting tableaux* if

1. there exists a content preserving bijection $\phi$ between the columns of $S_L$ and the columns of $S_R$;
2. for each simple column $c$ and any column $d$ in $S_L$, if $c$ appears to the left of $d$ in $S_L$ then $\phi(c)$ appears to the left of $\phi(d)$ in $S_R$.

## Proposition

*For any $w \in \mathbb{N}^*$, the meet-stalactic Q-symbol $(Q_{\mathrm{lSt}}(w), Q_{\mathrm{rSt}}(w))$ of $w$ is a pair of twin patience-sorting tableaux.*

# Robinson-Schensted-like

**Theorem**

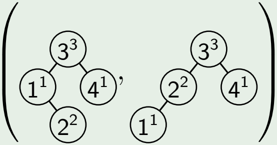
*The map* $w \mapsto (P_{\mathrm{mSt}}(w), Q_{\mathrm{mSt}}(w))$ *is a bijection between the elements of* $\mathbb{N}^*$ *and the set formed by the pairs* $\big((T_L, T_R), (S_L, S_R)\big)$ *where*

1. $(T_L, T_R)$ *is a pair of twin stalactic tableaux;*
2. $(S_L, S_R)$ *is a pair of twin patience-sorting tableaux with bijection* $\phi$*;*
3. $(T_L, T_R)$ *and* $(S_L, S_R)$ *have the same shape;*
4. *For each column* $c$ *in* $S_L$*, the column in* $T_L$ *in the same position as* $c$ *and the column in* $T_R$ *in the same position as* $\phi(c)$*, have the same content.*

We can equally define a $Q$-symbol for mTg by $Q_{\mathrm{mTg}}(w) = (P_{\mathrm{lTg}}(w), P_{\mathrm{rTg}}(w))$

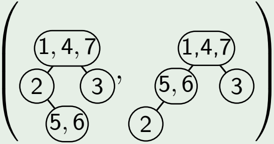
## Example

$P_{\mathrm{mTg}}(3143223) =$




and

$Q_{\mathrm{mTg}}(3143223) =$



# Twin binary search trees with multiplicity

## Definition

Let $T_L, T_R$ be BTSMs. We say $(T_L, T_R)$ is a *pair of twin binary trees with multiplicities* (pair of twin BTMs), if for all $i$,

1. $\mathrm{cont}(T_L) = \mathrm{cont}(T_R)$.
2. if the $i$-th node of $T_L$ has multiplicity 1 and has a left (resp. right) child then the $i$-th node of $T_R$ does not have a left (resp. right) child.

## Proposition

*For any $w \in \mathbb{N}^*$, the meet-taiga P-symbol $(P_{\mathrm{lTg}}(w), P_{\mathrm{rTg}}(w))$ of $w$ is a pair of twin BTMs.*

# Twin binary trees over sets

## Definition

Let $(S_L, S_R)$ be a pair of (respectively) increasing and decreasing BTSs. We say $(S_L, S_R)$ are a *pair of twin binary trees over sets* (pair of twin BTSs) if, for all $i$,

1. the $i$-th node of $S_L$ has the same label as the $i$-th node of $S_R$;
2. if the $i$-th node of $S_L$ is labelled by a set of cardinality 1 and has a left (resp. right) child, then the $i$-th node of $S_R$ does not have a left (resp. right) child.

## Proposition

*For any $w \in \mathbb{N}^*$, the meet-taiga Q-symbol $(Q_{\mathrm{lTg}}(w), Q_{\mathrm{rTg}}(w))$ of $w$ is a pair of twin BTSs.*

# Robinson-Schensted-like

**Theorem**

*The map $w \mapsto (P_{\mathrm{mTg}}(w), Q_{\mathrm{mTg}}(w))$ is a bijection between the elements of $\mathbb{N}^*$ and the set formed by the pairs $((T_L, T_R), (S_L, S_R))$ where*

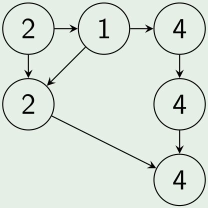
1. *$(T_L, T_R)$ is a pair of twin BSTMs;*
2. *$(S_L, S_R)$ is a pair of twin BTSs;*
3. *$(T_L, T_R)$ and $(S_L, S_R)$ have the same underlying pair of binary trees shape;*
4. *the multiplicity of the $i$-th node of $T_L$ (resp. $T_R$) is the cardinality of the set labelling the $i$-th node of $S_L$ (resp. $S_R$).*

**Example**

Consider the meet-stalactic tableau,

$$T = P_{\mathrm{mSt}}(214244) = \left( \begin{array}{|c|c|c|} \hline 2 & 1 & 4 \\ \hline 2 & & 4 \\ \cline{1-1} \cline{3-3} & & 4 \\ \cline{3-3} \end{array}, \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline & 2 & 4 \\ \cline{2-3} & & 4 \\ \cline{3-3} \end{array} \right)$$

Then, we define the following poset: $\psi_{\mathrm{mSt}}(T) =$



# Size of stalactic classes

## Proposition

*Let $w \in \mathbb{N}^*$. Then, each word in $[w]_{\mathsf{mSt}}$ is in bijection with a linear extension of $\psi_{\mathsf{mSt}}(P_{\mathsf{mSt}}(w))$.*

## Proposition

*There exists an algorithm to compute the number of linear extensions of $\psi_{\mathsf{mSt}}(T)$ has time complexity $\mathcal{O}(n^{2k-2}k!)$ where $n$ is the number of nodes and $k$ is the size of the support.*

## Theorem

*Let* $k \geq 2$, $V = (v_1, \ldots, v_k) \in \mathbb{N}_0^k$, $B = (b_1, \ldots, b_{k-1}) \in \mathbb{N}_0^{k-1}$ *and* $\sigma$ *be a permutation of* $[k]$. *Then, when* $\sigma_1 < \sigma_2$, $\mathcal{L}(G[V; B; \sigma])$ *is equal to*

$$\sum_{\substack{M \in \mathbb{N}_0^{\sigma_2 - \sigma_1} \\ 0 \leq \|M\|_1 \leq v_{\sigma_1}}} \mathcal{L}_M \cdot \binom{v_{\sigma_2} - 1 + v_{\sigma_1} - \|M\|_1}{v_{\sigma_1} - \|M\|_1} \prod_{i=1}^{\sigma_2 - \sigma_1} \binom{b_{\sigma_1 - 1 + i} + m_i}{m_i},$$

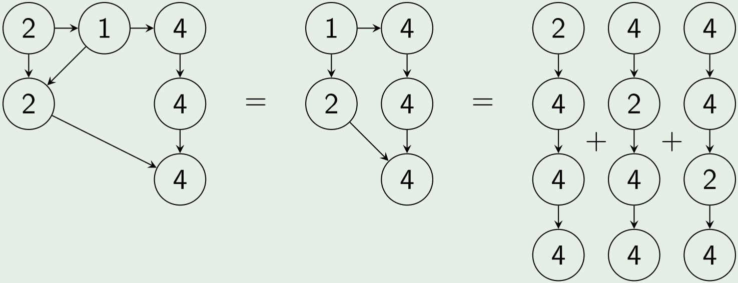
*where* $\mathcal{L}_M = 1$ *if* $k = 2$ *and* $\mathcal{L}_M = \mathcal{L}(G[f_{k,\sigma_1,\sigma_2,M}(V, B); \mathrm{Std}(\sigma_2\sigma_3 \cdots \sigma_k)])$ *otherwise, and, when* $\sigma_2 < \sigma_1$, *is equal to*

$$\sum_{\substack{M \in \mathbb{N}_0^{\sigma_1 - \sigma_2} \\ 0 \leq \|M\|_1 \leq v_{\sigma_2} - 1}} \mathcal{L}'_M \cdot \binom{v_{\sigma_1} + v_{\sigma_2} - 1 - \|M\|_1}{v_{\sigma_2} - 1 - \|M\|_1} \prod_{i=1}^{\sigma_1 - \sigma_2} \binom{b_{\sigma_2 - 1 + i} + m_i}{m_i},$$

*where* $\mathcal{L}'_M = 1$ *if* $k = 2$ *and* $\mathcal{L}'_M = \mathcal{L}(G[f_{k,\sigma_2,\sigma_1,M}(V, B); \mathrm{Std}(\sigma_1\sigma_3 \cdots \sigma_k)])$ *otherwise.*

# The Algorithm

## Example

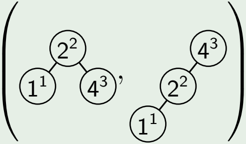


So, $[214244]_{\mathsf{mSt}} = 3$.

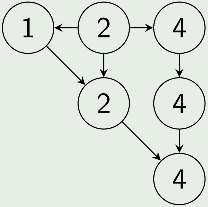
# Size of taiga classes

## Example

Consider the meet-taiga tableau,

$T = P_{\mathrm{mTg}}(214244) =$ 


Then, we define the following poset: $\psi_{\mathrm{mTg}}(T) =$



# Size of taiga classes

## Proposition

*Let $w \in \mathbb{N}^*$. Then, each word in $[w]_{\mathrm{mTg}}$ is in bijection with a linear extension of $\psi_{\mathrm{mTg}}(P_{\mathrm{mTg}}(w))$.*

## Theorem

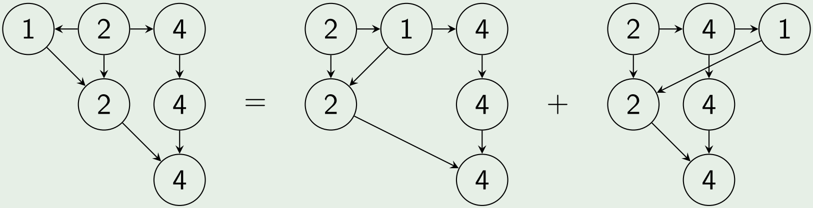
*Let $Q_{\mathrm{mTg}}(w) = T = (T_L, T_R)$. Then,*

$$\mathcal{L}(\psi_{\mathrm{mTg}}(T)) = \sum_{L} \sum_{R} \mathcal{L}(\mathrm{mSt}(P_{\mathrm{mSt}}(w_{L,R}))$$

*where the first sum is over all linear extensions $L$ of $\Delta(T_L)$ and the second sum is over all linear extensions $R$ of $\nabla(T_R, p_L)$ where $a < x$ in $p_L$ if and only if $a$ is simple in $(T_L, T_R)$ and $a < x$ in $L$.*

# The Algorithm

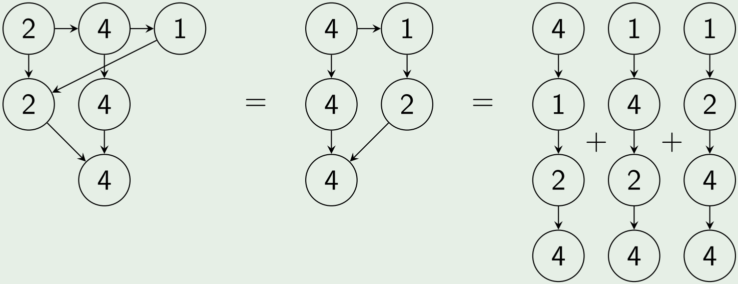
## Example




So, $[214244]_{\mathsf{mTg}} = [214244]_{\mathsf{mSt}} + [241244]_{\mathsf{mSt}}$.

# The Algorithm

## Example




So, $[214244]_{\mathrm{mTg}} = 3 + 3 = 6$.

# Size of taiga classes

**Proposition**

*There exists an algorithm to compute the number of linear extensions of* $\psi_{\mathsf{mTg}}(T)$ *has time complexity* $\mathcal{O}(n^{2k-2}(k!)^3)$ *where* $n$ *is the number of nodes and* $k$ *is the size of the support.*