Translational Hulls and Ideals

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Translations

Let S be a semigroup.

Definition

A *left translation* on S is a function $\lambda : S \longrightarrow S$ satisfying $\lambda(st) = \lambda(s)t$ for all $s, t \in S$.
A *right translation* on S is a function $\rho : S \longrightarrow S$ satisfying $(st)\rho = s(t)\rho$ for all $s, t \in S$.

The set of left [right] translations forms a monoid $\Lambda(S)$ [$P(S)$] under composition.

Definition

The *inner left [right] translations* are those induced by left [right] multiplication by an element s of S. We denote these by λ_s [ρ_s].

Translational Hulls

Definition

A left translation λ and right translation ρ are *linked* if $s\lambda(t) = (s)\rho t$ for any $s, t \in S$.

Definition

The subset of the direct product $\Lambda(S) \times P(S)$ consisting of linked translations forms a monoid $\Omega(S)$ called the *translational hull*. The elements of the translational hull are called *bitranslations*.

There is a *canonical homomorphism* $\pi : S \longrightarrow \Omega(S)$ given by

$$\pi(s) = (\lambda_s, \rho_s).$$

We call the image $\Pi(S)$ of π the *inner translational hull*.

More Translational Hulls

Definition

We say that S is *weakly reductive* if the canonical homomorphism π is injective.

Definition

A pair of bitranslations $(\lambda, \rho), (\lambda', \rho')$ are permutable if for all $s \in S$,

$$(\lambda(s))\rho' = \lambda((s)\rho')$$
$$(\lambda'(s))\rho = \lambda'((s)\rho).$$

If S is weakly reductive, then any pair of bitranslations of S are permutable.

The interest of the translational hull lies in the fact that any ideal extension of S canonically maps onto a semigroup of permutable bitranslations containing $\Pi(S)$.

Densely Embedded Ideals

Definition

An ideal I is *densely embedded* in S if

- any non-identity congruence σ on S restricts to a non-identity congruence $\sigma|_I$, and
- for any semigroup T properly containing S, there exists a non-identity congruence σ' such that $\sigma'|_I$ is the identity congruence

An ideal is densely embedded if and only if it is weakly reductive, with S canonically isomorphic to $\Omega(I)$.

Special Densely Embedded Ideals

Densely embedded ideals give an interesting way of detecting isomorphic semigroups.

If two semigroups have isomorphic densely embedded ideals, then the semigroups are isomorphic with each isomorphism of the ideals extending to a unique isomorphism of the semigroups.

Theorem (Gluskin)

Every non-zero ideal of $T_X, PT_X, B_X, \mathcal{I}_X$, or $\mathcal{O}_n$ is densely embedded.

Determining Sets

From now on S will be a monoid.

Definition

Let $A \subset I$. A is a *determining set* for $\Omega(I)$ if the map $\psi : (\lambda, \rho) \longrightarrow \lambda|_A$ is injective.

Lemma

Let A be a determining set for $\Omega(I)$. If the restriction map $\phi : \Lambda(S) \longrightarrow \Lambda(I)|_A$ is bijective then $\Omega(I) \cong S$.

Partition Monoids

Can Gluskin's results be extended to partition monoids?

No.
The analogous theorem does not hold for $\mathcal{P}_n, \mathcal{B}_n, \mathcal{PB}_n, \mathcal{J}_n$, or $\mathcal{M}_n$.

The rank 1 ideals are also not densely embedded.

Thanks

References

Mario Petrich. "The Translational Hull in Semigroups and Rings". In: *Semigroup Forum* 1 (1970), pp. 283–360.