Ends for Monoids and Semigroups

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Main Results

- If G is finitely generated infinite group, then the number of ends of G is **1,2** or ∞.
  If **H** is a subgroup of finite index in **G** then **G** and **H** have the same number of ends.
  (Cohen [2], Dunwoody[3], Schupp [15], Stallings [18, 19])
- For direct products and for many other semidirect products of finitely generated infinite monoids, the right Cayley digraph of the semidirect product has **1** end.
  For a finitely generated subsemigroup of a free semigroup the number of ends is **1** or ∞.

Basic Definitions

- **Graph** $\Gamma = (V, E, \iota, \tau, {}^{-1})$
- **Digraph** $\Gamma = (V, E, \iota, \tau)$
- For $\mathfrak{F}$ a subset of V, we write $\Gamma - \mathfrak{F}$ for the **full subgraph** of Γ on $V - \mathfrak{F}$.
- **Functor** from $\Gamma = (V, E, \iota, \tau)$ to $\widehat{\Gamma} = (V, E \cup E^{-1}, \iota, \tau, {}^{-1})$

Walks, Paths, Geodesics

- A (positive) **walk** ω of length n is a sequence $(e_1, e_2, \ldots, e_n)$ such that $\tau(e_i) = \iota(e_{i+1})$ for $1 \leq i < n$. (We often write $\omega = e_1 e_2 \ldots e_n$)
- A walk is a **path** if all its vertices are distinct.
- The **distance**, $d_\Gamma(v_1, v_2)$, between v_1 and v_2 in Γ, is the length of the shortest path in Γ from v_1 to v_2.
- A (positive) path of minimal length from v_1 to v_2 in Γ is a (di)**geodesic** in Γ.

Unbounded Paths and Infinite Components

- A graph Γ has **unbounded paths (unbounded geodesics)** if for every natural number n there is a path (geodesic) of length n in Γ.
- A graph Γ is **connected** if there is a path in Γ from any vertex v_1 to any vertex v_2. We will define a **digraph** Γ to be **connected** if $\widehat{\Gamma}$ is connected.
- A **component** of a graph or of a digraph Γ is a maximal connected subgraph of Γ.

Number of Ends of a Graph

- For Γ, a graph (digraph) and $\mathfrak{F}$, a finite set of vertices of Γ, we define for various subscripts x, $\mathfrak{C}_x(\Gamma - \mathfrak{F})$ a **set of "infinite" components** of $\Gamma - \mathfrak{F}$.
- For each subscript **x**, we will define $e_x(\Gamma)$, **a number of ends of Γ** by

$$e_x(\Gamma) = \sup_{\mathfrak{F} \subseteq V,\ \mathfrak{F} \text{ finite}} |\mathfrak{C}_x(\Gamma - \mathfrak{F})|.$$

- There are numerous equivalent definitions for the number of **ends for a finitely generated group** (Cohen [2], Dunwoody[3], Schupp [15], Stallings [18, 19]).

Variations for Number of Ends

- $\mathfrak{C}_\infty(\Gamma) =$
  $\{C : C$ is a component of Γ having **infinitely many vertices**$\}$
- $\mathfrak{C}_p(\Gamma) =$
  $\{C : C$ is a component of Γ having **unbounded paths**$\}$
- $\mathfrak{C}_g(\Gamma) =$
  $\{C : C$ is a component of Γ having **unbounded geodesics**$\}$
- $\mathfrak{C}_*(\Gamma) =$
  $\{C : C$ contains a vertex which **initiates unbounded paths**$\}$
- $\mathfrak{C}_\dagger(\Gamma) =$
  $\{C :$
  C contains a vertex that **initiates unbounded geodesics**$\}$

Example 1

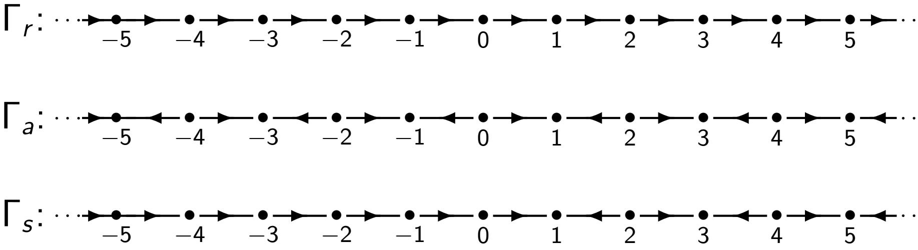


Figure: Γ_r, Γ_a and Γ_s

Example 1

Example 1

- $\widehat{\Gamma}_r = \widehat{\Gamma}_a = \widehat{\Gamma}_s$, so that $e_x(\Gamma_r) = e_x(\Gamma_a) = e_x(\Gamma_s) = \mathbf{2}$ for any **graph** subscript **x**.
- For Γ_r, we observe that $e_{+p}(\Gamma_r) = e_\delta(\Gamma_r) = \mathbf{2}$, while $e_{\overrightarrow{*}}(\Gamma_r) = e_{\overrightarrow{\delta}}(\Gamma_r) = e_{\overleftarrow{*}}(\Gamma_r) = e_{\overleftarrow{\delta}}(\Gamma_r) = \mathbf{1}$.
- Since no positive path in Γ_a has length greater than 1, $e_x(\Gamma_a) = \mathbf{0}$ for every digraph subscript **x**.
- Similarly, $e_{+p}(\Gamma_s) = e_\delta(\Gamma_s) = e_{\overleftarrow{*}}(\Gamma_s) = e_{\overleftarrow{\delta}}(\Gamma_s) = \mathbf{1}$, while $e_{\overrightarrow{*}}(\Gamma_s) = e_{\overrightarrow{\delta}}(\Gamma_s) = \mathbf{0}$.

Graph and Digraph Definitions of Ends

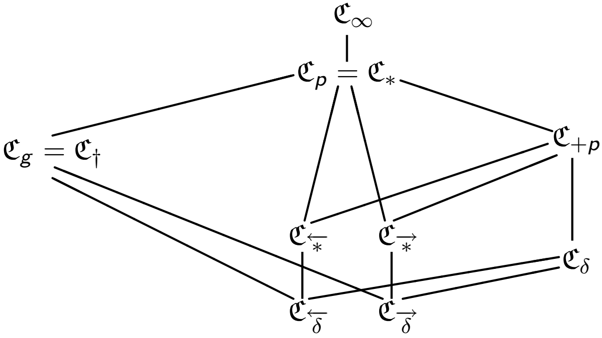


Figure: Some subset inclusions for $\mathfrak{C}_\infty$

- Cayley graphs of groups are a fundamental tool in combinatorial group theory (see Lyndon and Schupp [10] and Magnus, Karrass, and Solitar [11]).
- Cayley graphs of groups represent a link between topology, graph theory, and automata theory.
- Combinatorial properties of Cayley graphs of monoids were studied by Zelinka [20] and by Kelarev, Praeger, and Quinn in [6, 7, 8]
- Cayley graphs of automatic monoids were studied by Silva and Steinberg in [16, 17]
- Logical aspects of Cayley graphs of monoids were studied by Kuske and Lohrey in [9]

Right and Left Cayley Digraphs

T a semigroup and $X \subseteq T$ a set of semigroup generators for T

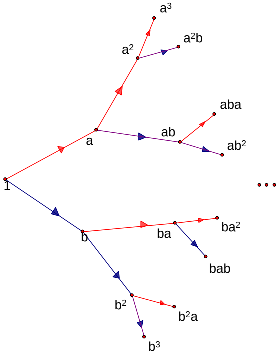
- The **right Cayley digraph** for T with respect to X is the digraph $\Gamma_r(T, X) = (V, E, \iota, \tau)$ where $V = T$, $E = T \times X = \{(t, x) : t \in T, x \in X\}$, $\iota((t, x)) = t$ and $\tau((t, x)) = tx$.

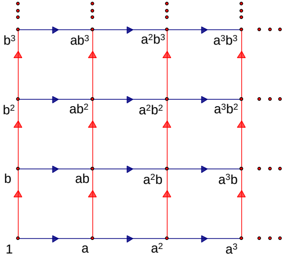
$$t \xrightarrow{x} tx$$

- the **left Cayley digraph** for T with respect to X is the digraph ${}_\ell\Gamma(X, T) = (V, E, \iota, \tau)$ where $V = T$, $E = X \times T = \{(x, t) : x \in X, t \in T\}$ $\iota((x, t)) = t$ and $\tau((x, t)) = xt$.

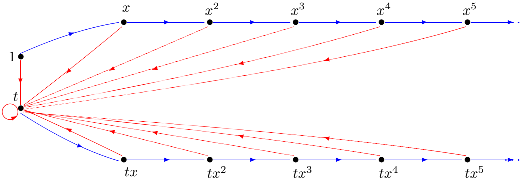
$$t \xrightarrow{x} xt$$

Right Cayley Digraphs for the Free Monoid $F(a,b)$ and the Free Commutative Monoid $M = \langle a, b : ab = ba \rangle$

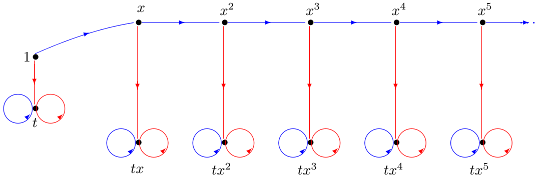




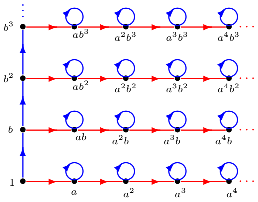
Right Cayley Digraph for $M = \langle x, t : xt = t, t^2 = t \rangle$



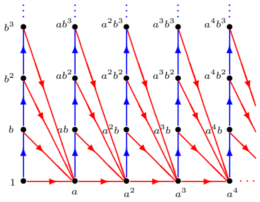
Left Cayley Digraph for $M = \langle x, t : xt = t, t^2 = t \rangle$



Left and Right Cayley Digraphs for $M = \langle a, b : ba = a \rangle$

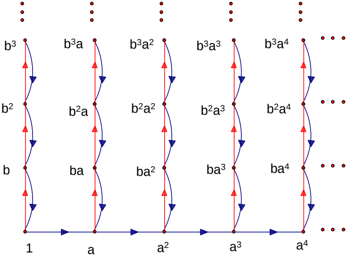


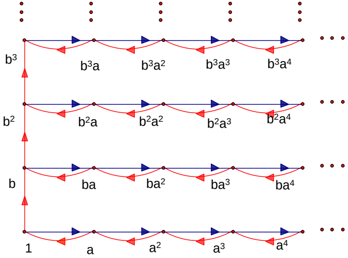
Left digraph



Right digraph

Left and Right Cayley Digraphs for Bicyclic Monoid $M = \langle a, b : ab = 1 \rangle$





- Lemma 2
  Let X be a finite set of monoid generators for the monoid M and Γ be the right Cayley digraph, $\Gamma_r(M, X)$. If $\mathfrak{F}$ is any finite set of vertices of Γ and **C** *is an* **infinite component** *of* $\mathbf{\Gamma} - \mathfrak{F}$, *then there is a* **vertex** $\hat{\mathbf{v}}$ *in* **C** *which* **initiates unbounded digeodesics**.

- Corollary 3
  $e_{\mathbf{x}}(\Gamma) = e_{\infty}(\Gamma)$ *if* $\mathbf{x} \in \{p, g, *, \dagger, +p, \delta, \overrightarrow{*}, \overrightarrow{\delta}\}$.

- Lemma 4
  For a monoid M and its finite subset $\mathfrak{F}$, $\Gamma - \mathfrak{F}$ has at most $1 + |X|\,|\mathfrak{F}|$ components.
  - FACTS:
    - $e_\infty(\Gamma) \geq 1$ for infinite monoids.
    - Let $\mathfrak{F}$ and $\hat{\mathfrak{F}}$ be finite subsets of M with $\mathfrak{F} \subseteqq \hat{\mathfrak{F}}$. Then $|\mathfrak{C}_\infty(\Gamma - \mathfrak{F})| \leq |\mathfrak{C}_\infty(\Gamma - \hat{\mathfrak{F}})|$.
    - For every natural number n, define $\mathfrak{F}_n$ to be $\{m \in M : L_X(m) \leq n\}$. Then $\mathfrak{F}_n$ is finite and $e_\infty(\Gamma) = \lim_{n\to\infty} |\mathfrak{C}_\infty(\Gamma - \mathfrak{F}_n)|$.

Ends are Independent of the Set of Generators

Lemma 5

If X and Y are finite sets of monoid generators for the monoid M, then $e_\infty(\Gamma_r(M, X)) = e_\infty(\Gamma_r(M, Y))$ and $e_\infty({}_\ell\Gamma(X, M)) = e_\infty({}_\ell\Gamma(Y, M))$.

Proof.

- It suffices to prove that $e_\infty(\Gamma_r(M, X)) = e_\infty(\Gamma_r(M, X \cup Y))$
- Reduce to the case that $e_\infty(\Gamma_r(M, X)) = e_\infty(\Gamma_r(M, X \cup \{y\}))$ where $y \in Y$ by using induction on $|X \cup Y| - |X|$. For brevity, write $\Gamma = \Gamma_r(S, X)$ and $\Gamma' = \Gamma_r(S, X \cup \{y\})$.
- We consider two cases, when $e_\infty(\Gamma)$ is finite or infinite.
- We first show $e_\infty(\Gamma) \leq e_\infty(\Gamma')$ in the finite case.

□

Continuation of the Proof that Ends are Independent of the Set of Generators

- Second, we exhibit a finite set $\mathfrak{F}_2$ such that $\Gamma' - \mathfrak{F}_2$ has $e_\infty(\Gamma)$ infinite components, proving the equality in the finite case.
- Last, when $e_\infty(\Gamma)$ is infinite, we show that for any natural number n, there is a finite subset $\mathfrak{F}$ of M such that $\Gamma' - \mathfrak{F}$ has at least n infinite components.

- Definition 6
  For a finitely generated semigroup S, we define $\mathcal{E}_{\mathbf{r}}(\mathbf{S})$ and $\mathcal{E}_{\ell}(\mathbf{S})$ by $\mathcal{E}_{\mathbf{r}}(\mathbf{S}) = \mathbf{e}_{\infty}(\Gamma_{\mathbf{r}}(\mathbf{S}, \mathbf{X}))$ and $\mathcal{E}_{\ell}(\mathbf{S}) = \mathbf{e}_{\infty}({}_{\ell}\Gamma(\mathbf{S}, \mathbf{X}))$ for any finite set $\mathbf{X}$ of semigroup generators for $\mathbf{S}$.
  - When M is a finitely generated monoid, the values for $\mathcal{E}_r(M)$ and $\mathcal{E}_\ell(M)$ do not change if we consider M as a semigroup rather than as a monoid.
  - It is usual to consider a Cayley graph rather than a Cayley digraph for a group. Typically, these are the right Cayley graphs (isomorphic to the left Cayley graphs)which are always locally finite.
  - If a group is considered as a monoid , then its number of ends (considered as a group) is equal to both of the monoid values $\mathcal{E}_r(G)$ and $\mathcal{E}_\ell(G)$.

- Definition 7
  For any semigroup $(S,\cdot)$ the dual semigroup $S^{\mathrm{op}}=(S,*)$ has the same set of elements as S and has multiplication $*$ defined by $s_1 * s_2 = s_2 \cdot s_1$.
- Dual Semigroup Proposition
  If the semigroup S is isomorphic to S^{op}, then $\mathcal{E}_r(S)=\mathcal{E}_\ell(S)$.
- Corollary 8
  If T is a finitely generated **inverse** *semigroup (or a finitely generated inverse monoid), then $\mathcal{E}_r(T)=\mathcal{E}_\ell(T)$.*

Special Semidirect Product of Monoids

- $M = \langle X : R_2 \rangle$ and $T = \langle A : R_1 \rangle$ are monoids
- Define $\boldsymbol{\theta}_{\mathbf{T}} \in \mathrm{End}(T)$ as $\theta_T(t) = 1_T, t \in T$ and $\boldsymbol{\iota}_{\mathbf{T}}$ as the identity automorphism of T
- $\Phi_0 : M \to \mathrm{End}(T)$ takes 1_M to ι_T and every other element of M to θ_T.
- $\hat{M} = T \rtimes_{\Phi_0} M = \langle A \cup X \ : \ R_1 \cup R_2 \cup \{(xa, x) : a \in A, \ x \in X\} \rangle$

- Layer Lemma

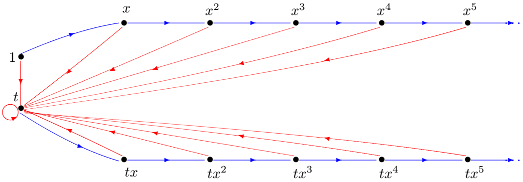
  Let T be a **finite** *monoid and M a finitely generated monoid. Assume that* $\mathbf{M} = \mathbf{S^1}$ *for some semigroup S. Then $\mathcal{E}_r(T \rtimes_{\Phi_0} M) = |T| \mathcal{E}_r(M)$ and $\mathcal{E}_\ell(T \rtimes_{\Phi_0} M) = \mathcal{E}_\ell(M)$.*

Number of Ends for $A_n = \langle x, t : xt = t, t^n = t^{n-1} \rangle = T \rtimes_{\Phi_0} M$

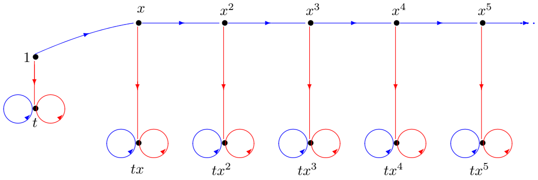
Example 9

- T is monogenic monoid with presentation $T = \langle t : t^n = t^{n-1} \rangle$
- $M = S^1$ is infinite monogenic monoid whose left and right Cayley digraphs have **1** end
- By Layer Lemma, $\mathcal{E}_r(A_n) = |T|\mathcal{E}_r(M) = n \cdot 1 = \mathbf{n}$ and $\mathcal{E}_\ell(A_n) = \mathcal{E}_\ell(M) = \mathbf{1}$

Right Cayley Digraph for $A_2 = \langle x, t : xt = t, t^2 = t \rangle$ with 2 Ends



Left Cayley Digraph for $A_2 = \langle x, t : xt = t, t^2 = t \rangle$ with **1** End



Number of Ends for $J_{n,m} = T \rtimes_{\Phi_0} A_n^{\text{op}}$

Example 10

- T is monogenic monoid of order m
- $A_n^{\text{op}} = S^1$ is infinite monogenic monoid whose left Cayley graph has **n** ends and right Cayley digraphs has **1** end
- By Layer Lemma, $\mathcal{E}_r(T \rtimes_{\Phi_0} A_n^{\text{op}}) = m \cdot \mathcal{E}_r(A_n^{\text{op}}) = m \cdot \mathcal{E}_\ell(A_n) =$**m** and
- $\mathcal{E}_\ell(T \rtimes_{\Phi_0} A_n^{\text{op}}) = \mathcal{E}_\ell(A_n^{\text{op}}) = \mathcal{E}_r(A_n) =$**n**

Special Semidirect Products

- Write Monic(M) for the submonoid of End(M) consisting of one-to-one endomorphisms.
- Write $\mathrm{End}_r(M)$ for End(M) when functions act on their arguments from right and $\mathrm{End}_\ell(M)$ when functions act on their arguments from left.
- If $\Phi : A \to \mathrm{End}_r(B)$ is a monoid homomorphism, define the monoid semi-direct product $A \ltimes_\Phi B$ to have elements $\{(a, b) \ : \ a \in A, b \in B\}$ and multiplication $(a_1, b_1)(a_2, b_2) = (a_1 a_2, b_1^{a_2} b_2)$.
- Similarly, if $\Phi : A \to \mathrm{End}_\ell(B)$ is a monoid homomorphism, we define the monoid semi-direct product $B \rtimes_\Phi A$ to have elements $\{(b, a) \ : \ b \in B, a \in A\}$ and multiplication $(b_1, a_1)(b_2, a_2) = ((b_1)({}^{a_1} b_2), a_1 a_2)$.

Special Semidirect Products

- Theorem 11
  Suppose that M_i is a finitely generated infinite monoid for $i = 1, 2$. If $\Phi : M_1 \to \mathrm{Monic}(M_2)$ is a monoid homomorphism, then $\mathcal{E}_r(M_1 \ltimes_\Phi M_2) = \mathcal{E}_\ell(M_2 \rtimes_\Phi M_1) = 1$.

- Corollary 12
  Suppose that G_i is a finitely generated infinite group for $i = 1, 2$. If $\Phi : G_1 \to \mathrm{Aut}(G_2)$ is a group automorphism, then the group semidirect product $G_2 \rtimes_\Phi G_1$ has one end.

Special Semidirect Products

- Corollary 13

  Suppose, for $i = 1, 2$, that M_i is an infinite monoid with a finite set of monoid generators X_i. Let $M = M_1 \times M_2$ be the monoid direct product. Then $\mathcal{E}_r(M) = \mathcal{E}_\ell(M) = 1$.

- Proof.

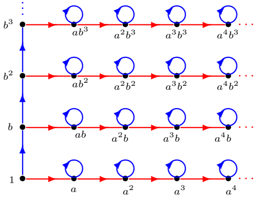
  The direct product is a special case of Theorem 11 where Φ takes each element of M_1 to the identity automorphism of M_2. □

$M = \langle a, b : ba = a \rangle = B \rtimes_{\Phi_0} A$

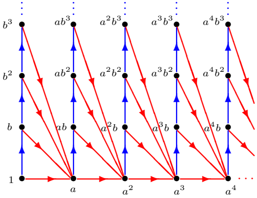
Example 14

- In the previous theorem, the hypothesis that Φ has its range in Monic($\mathbf{M_2}$) rather than just in End(M_2) is necessary.
- $\mathcal{E}_\ell(B \rtimes_{\Phi_0} A) = \mathcal{E}_\ell(A)$ of the Layer Lemma need not hold if B is an **infinite** monoid.
- Let $A = \langle a \rangle$ and $B = \langle b \rangle$ be free monogenic monoids and $M = A \ltimes_{\Phi_0} B$.
- Here $a\Phi_0 = \theta_B$ where $b^m\theta_B = 1_B$ for every non-negative integer m, hence θ_B is **not one-to-one**.
- $$\mathcal{E}_r(A \ltimes_{\Phi_0} B) = \mathcal{E}_\ell(A \ltimes_{\Phi_0} B) = \mathcal{E}_r(B \rtimes_{\Phi_0} A) = \mathcal{E}_\ell(B \rtimes_{\Phi_0} A) = \infty.$$

Left and Right Cayley Digraphs for $M = \langle a, b : ba = a \rangle$



Left digraph



Right digraph

0-direct Unions

- Let Λ be an index set and $(S_\lambda, *_\lambda)$ be a semigroup for each $\lambda \in \Lambda$. Assume that $S_{\lambda_1} \cap S_{\lambda_2} = \emptyset$ if $\lambda_1 \neq \lambda_2$ and that 0 is a new element not in $\cup S_\lambda$. Define $\vee S_\lambda$ to be $\{0\} \cup \left(\bigcup_{\lambda \in \Lambda} S_\lambda\right)$ and define a multiplication $*$ on $\vee S_\lambda$ by

$$s*t = \begin{cases} s *_\lambda t & \text{if there exists } \lambda \in \Lambda \text{ such that } s \in S_\lambda \text{ and } t \in S_\lambda \\ 0 & \text{otherwise} \end{cases}.$$

- For any λ, define S_λ^0 to be the semigroup having elements $\{0\} \cup S_\lambda$ with the multiplication $*_\lambda$ extended by setting $s *_\lambda 0 = 0 *_\lambda s = 0 *_\lambda 0 = 0$ for all $s \in S_\lambda$. Then $\vee S_\lambda$ is the 0-direct union of the semigroups S_λ^0. See Clifford and Preston [1, Volume II, page 13], Howie, [5, page 71] or Higgins, [4, page 26].

0-direct Unions

- Lemma 15
  Suppose that Λ is a finite set and that $\{S_\lambda\}_{\lambda\in\Lambda}$ is a set of pairwise disjoint, finitely generated semigroups S_λ. Then $\vee S_\lambda$ is finitely generated, $\mathcal{E}_\ell(\vee S_\lambda) = \Sigma_{\lambda\in\Lambda}\mathcal{E}_\ell(S_\lambda)$ and $\mathcal{E}_r(\vee S_\lambda) = \Sigma_{\lambda\in\Lambda}\mathcal{E}_r(S_\lambda)$.

- Example 16

  For an arbitrary natural number n, let Λ be an index set with $|\Lambda| = n$ and for each $\lambda \in \Lambda$, let S_λ be a finitely generated abelian group with $\mathcal{E}_\ell(S_\Lambda) = \mathcal{E}_r(S_\Lambda) = 1$. For example, take S_λ to be the free abelian group of rank $r_\lambda \geq 2$. Let $S = \vee S_\lambda$. Then S is a finitely generated, completely regular, commutative inverse semigroup with $\mathcal{E}_r(S) = \mathcal{E}_\ell(S) = n$.

Ends for the additive semigroup $\mathbb{N}$ of natural numbers

- The group versions of the following theorem in Lyndon and Schupp [10, Proposition I.2.17] and Magnus, Karrass, and Solitar [11, Exercise 1.4.6] are easily modified to obtain the semigroup version.

- Lyndon's Theorem

  (Mateescu and Salomaa[12, Theorem 2.2]) Suppose that F is the free semigroup on the alphabet A and that $u, v \in F$. If $uv = vu$, then there is an element $w \in F$ and natural numbers m, n such that $u = w^m$ and $v = w^n$.

- Lemma 17

  If S is any subsemigroup of the additive semigroup $\mathbb{N}$ of natural numbers, then $\mathcal{E}_\ell(S) = \mathcal{E}_r(S) = 1$.

Proof that subsemigroups of additive semigroup $\mathbb{N}$ have one end:

- Let S be a subsemigroup of the additive semigroup $\mathbb{N}$. Since S is commutative, from Dual Semigroup Proposition we must have $\mathcal{E}_\ell(S) = \mathcal{E}_r(S)$.
- From elementary number theory we know that S contains all but finitely many natural numbers.
- Write $n_0 - 1$ for the greatest natural number that is not in S. Then $S = X_0 \cup \{n \in \mathbb{N} : n \geq n_0\}$ for some finite set $X_0 \subseteq \mathbb{N}$.
- S is generated by the finite set $X = X_0 \cup \{n \in \mathbb{N} : n_0 \leq n < 2n_0\}$.

Continuation of the proof that subsemigroups of additive semigroup $\mathbb{N}$ have one end:

- Write Γ for $\Gamma_r(S, X)$ and $\mathfrak{F}$ for any finite subset of vertices of Γ.
- Let m be the largest element in $\mathfrak{F}$ and choose $k \in \mathbb{N}$ which satisfies $m < kn_0$.
- $C = \{n : n \geq (k+1)n_0\}$ is an infinite subset of $\Gamma - \mathfrak{F}$ having a finite complement in $\mathbb{N}$.
- To prove that Γ has only one end, it suffices to show that C is a subset of the component of $\Gamma - \mathfrak{F}$ which contains kn_0.
- $$kn_0 \xrightarrow{n_0} (k+1)n_0 \xrightarrow{n_0} (k+2)n_0 \quad \ldots$$
$$\ldots \quad \xrightarrow{n_0} (q-1)n_0 \xrightarrow{n_0+r} qn_0 + r$$

- Theorem 18
  If S is a commutative subsemigroup of a free semigroup, then $\mathcal{E}_\ell(S) = \mathcal{E}_r(S) = 1$.

- Lemma 19
  Let F be the free semigroup on the alphabet A and let S be a finitely generated subsemigroup of F with finite set of generators X. Let Γ be the right Cayley graph $\Gamma_r(S, X)$. If $\mathfrak{F}$ is a finite subset of S and w is a element of $S - \mathfrak{F}$, write C_w for the component of $\Gamma - \mathfrak{F}$ containing w. If the length, $L_A(w)$, of w on the alphabet A is minimal among elements of $S - \mathfrak{F}$, then w is a prefix of every vertex in C_w.

Non Commutative Subsemigroups and Monoids

- Theorem 20
  If S is a finitely generated subsemigroup of a free semigroup and S is not commutative, then $\mathcal{E}_\ell(S) = \mathcal{E}_r(S) = \infty$.
  - The analogous results for submonoids of free monoids follow immediately by adjoining the empty word.

Questions for Further Consideration

- A finitely generated group has 1, 2, or ∞ many ends. What can we say about number of ends of right cancellative semigroups (whose Cayley graphs are locally finite)?
- Subgroups of finite index of f.g. groups have the same number of ends.
- (R. Gray) Do f.g. submonoid with a finite Rees index in a f.g. monoid and that monoid have the same number of ends?
- What can we say about ends for Schtzenberger graphs of f.g. inverse monoids?

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