Homotopical and homological finiteness properties of monoids and their subgroups

Robert Gray

University of St Andrews

NBSAN
St Andrews, April 2009

Engineering and Physical Sciences Research Council

Why are fractions so hard?

$$\frac{297}{198} \qquad \frac{147}{98} \qquad \frac{152}{95}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\frac{3}{2} = \frac{3}{2} \neq \frac{8}{5}$$

Question: Why are presentations so hard?

Answer: Because they are harder than fractions!

Presentations

Fact: There are lots of *nasty* finitely presented monoids out there.

Markov (1947), Post (1947): There exist finitely presented monoids for which there is no algorithm to solve the word problem.

Idea

1. Identify a class $\mathcal{C}$ of "nice" finite presentations:
   - finite complete rewriting systems
2. Try to gain understanding of those monoids that may be defined by presentations from $\mathcal{C}$:
   - study properties of monoids defined by such rewriting systems:
     - Finite derivation type (FDT)
     - FP_n

Rewriting systems

- A - non-empty set (the alphabet), A^* - free monoid over A
- A rewriting system over A is a subset $R \subseteq A^* \times A^*$
- Rewrite rules: $(r_{+1}, r_{-1}) \in R$, also written as $r_{+1} = r_{-1}$.
- Write $u \to_R v$ if $u \equiv w_1 r_{+1} w_2$ and $v \equiv w_1 r_{-1} w_2$ where $(r_{+1}, r_{-1}) \in R$ and $w_1, w_2 \in A^*$.
- $\to_R^* =$ the reflexive transitive closure of $\to_R$

$$\begin{aligned} \leftrightarrow_R^* &= \text{the reflexive symmetric transitive closure of } \to_R \\ &= \text{the congruence on } A^* \text{ generated by } R \end{aligned}$$

- $\langle A | R \rangle$ - monoid presentation with generators A and set of defining relations R
- $A^* / \leftrightarrow_R^*$ - the monoid defined by the presentation $\langle A | R \rangle$
- A rewriting system (presentation) is called finite if both A and R are finite.

Noetherian rewriting systems

- R - a rewriting system on A

Definition

We say that R is noetherian if there is no infinite sequence

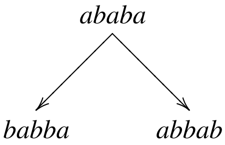
$$w_1 \to_R w_2 \to_R w_3 \to_R \cdots$$

- A word $w \in A^*$ is called irreducible if there does not exist any word $v \in A^*$ such that $w \to_R v$.
- If R is noetherian then for any $w \in A^*$ we can obtain an irreducible $\hat{w}$ with $w \to_R^* \hat{w}$.

Example

Kupar and Narendran (1985)

- $\mathcal{P}_1 = \langle a, b | aba = bab \rangle$



If $w_1 \rightarrow_R w_2$ then

- w_2 has strictly more bs than w_1 and;
- w_2 and w_1 have the same length.

There are only finitely many words over $\{a, b\}$ of any given fixed length, so there is no infinite sequence

$$w_1 \rightarrow_R w_2 \rightarrow_R w_3 \rightarrow_R \cdots$$

Hence $\mathcal{P}_1$ is Noetherian.

Complete rewriting systems

- R - a rewriting system on A

Definition

We say that R is a complete rewriting system if R is noetherian and every $\leftrightarrow_R^*$-class contains exactly one irreducible word.

The word problem

Definition

A monoid M with a finite generating set A has soluble word problem if there is an algorithm which for any two words $w_1, w_2 \in A^*$ decides whether or not they represent the same element of M.

Proposition

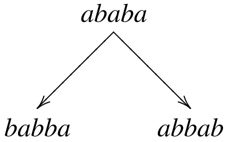
If M is presented by a finite complete rewriting system then M has soluble word problem.

- **Normal form algorithm:** given $u, v \in A^*$, reduce $u \to^* u_0$ and $v \to^* v_0$ to irreducible words u_0 and v_0, then check if $u_0 \equiv v_0$ in A^*.

Example
Kupar and Narendran (1985)

- $\mathcal{P}_1 = \langle a, b | aba = bab \rangle$

Is **not** a complete rewriting system since irreducibles not unique



- $\mathcal{P}_2 = \langle a, b, c | ab = c, ca = bc, bcb = cc, ccb = acc \rangle$
- $\mathcal{P}_2$ **is** a complete rewriting system.
- $\mathcal{P}_1$ and $\mathcal{P}_2$ define the same monoid.

Question

Which monoids can be presented by finite complete rewriting systems?

Finite derivation type
a homotopical finiteness condition

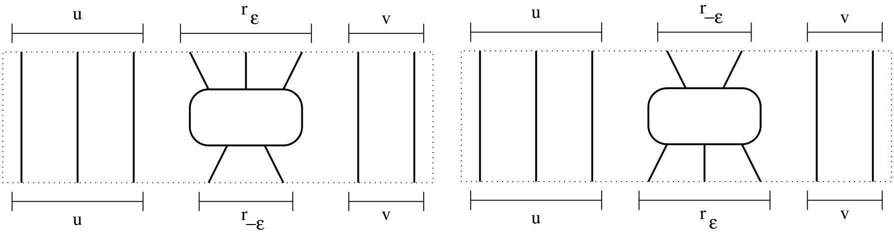
- Is a property of finitely presented monoids.
- Introduced by **Squier (1994)**.

Original motivation

To capture much of the information of a finite complete rewriting system for a monoid in a property which is independent of the choice of presentation.

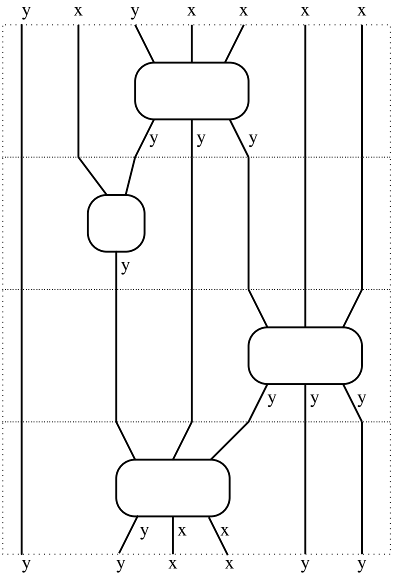
The derivation graph of a presentation

- $\mathcal{P} = \langle A | R \rangle$ a monoid presentation
- Derivation graph: $\Gamma = \Gamma(\mathcal{P}) = (V, E, \iota, \tau, ^{-1})$:
  - Vertices: $V = A^*$
  - Edges are 4-tuples: $\{(u, r, \epsilon, v) : u, v \in A^*, r = (r_{+1}, r_{-1}) \in R, \text{ and } \epsilon \in \{+1, -1\}\}$.
- Initial and terminal vertices: $\iota, \tau : E \to V$ for $\mathbb{E} = (u, r, \epsilon, v)$ (with $r = (r_{+1}, r_{-1}) \in R$):
  - $\iota\mathbb{E} = ur_\epsilon v$
  - $\tau\mathbb{E} = ur_{-\epsilon} v$
- Inverse edge mapping: $^{-1} : E \to E$
  - $(u, r, \epsilon, v)^{-1} = (u, r, -\epsilon, v)$.



$\mathbb{E} = (u, r, \epsilon, v)$ $\qquad$ $\mathbb{E}^{-1} = (u, r, -\epsilon, v)$

Paths and pictures



Example. $\langle x, y | \underbrace{xy = y}_{r}, \underbrace{yx^2 = y^3}_{s} \rangle$

A path is a sequence $\mathbb{P} = \mathbb{E}_1 \circ \mathbb{E}_2 \circ \ldots \circ \mathbb{E}_n$ where $\tau\mathbb{E}_i \equiv \iota\mathbb{E}_{i+1}$.

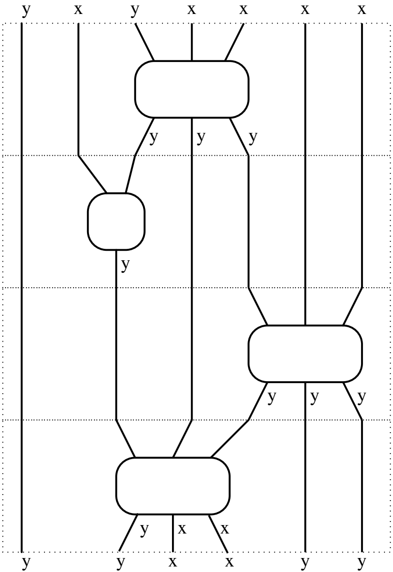
Gluing edge-pictures together we obtain pictures for paths.

ι and τ can be defined for paths

In this example
$\iota\mathbb{P} = yxyxxxx, \ \tau\mathbb{P} = yyxxyy.$

Paths and pictures

Example. $\langle x, y | \underbrace{xy = y}_{r}, \underbrace{yx^2 = y^3}_{s} \rangle$



$(yx, s, +1, x^2)$

$(y, r, +1, y^2x^2)$

$(y^3, s, +1, 1)$

$(y, s, -1, y^2)$

Operations on pictures

$\mathcal{P} = \langle A|R \rangle, \quad \Gamma = \Gamma(\mathcal{P})$

$$\text{Pictures} \leftrightsquigarrow \text{Paths}$$

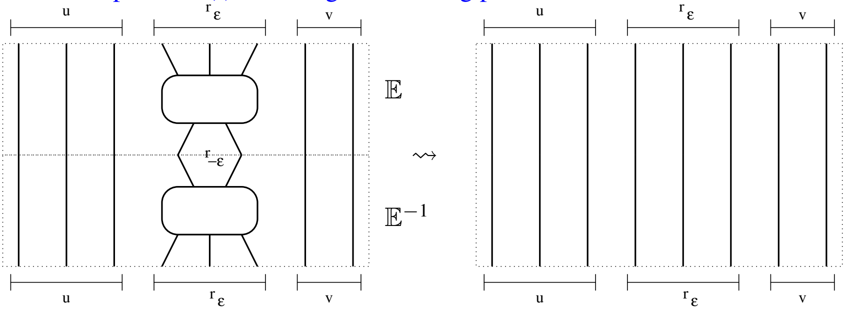
- $P(\Gamma)$ - of all paths in Γ
- Parallel paths: write $\mathbb{P} \parallel \mathbb{Q}$ if $\iota\mathbb{P} \equiv \iota\mathbb{Q}$ and $\tau\mathbb{P} \equiv \tau\mathbb{Q}$.
- $\parallel \subseteq P(\Gamma) \times P(\Gamma)$ - the set of all parallel paths
- **X** - set of pairs of paths $(\mathbb{P}_1, \mathbb{P}_2)$ such that $\mathbb{P}_1 \parallel \mathbb{P}_1$

Idea

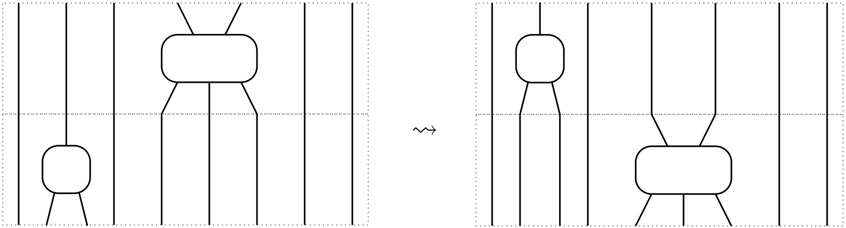
Want to regard certain paths as being equivalent to one another modulo **X**.

Operations on pictures

Basic operation (I): Deleting a cancelling pair



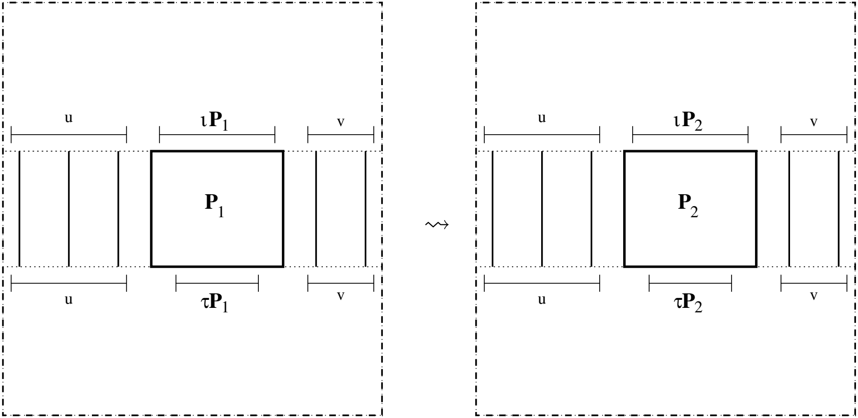
Basic operation (II): Interchanging disjoint discs



Operations on pictures

Basic operation (III): Replacing a subpicture using **X**

Replace a subpicture $\mathbb{P}_1$ by $\mathbb{P}_2$ provided $(\mathbb{P}_1, \mathbb{P}_2) \in \mathbf{X}$.



$$u \cdot \mathbb{P}_1 \cdot v \quad \rightsquigarrow \quad u \cdot \mathbb{P}_2 \cdot v$$

Homotopy bases

Note: Applications of these picture operations do not change the initial vertex or the terminal vertex of the original path.

A homotopy base is...

a set $\mathbf{X}$ of parallel paths such that given an arbitrary pair $(\mathbb{P}_1, \mathbb{P}_2) \in \parallel$ we can transform $\mathbb{P}_1$ into $\mathbb{P}_2$ by a finite sequence of elementary picture operations (and their inverses)

(I) cancelling pairs, (II) disjoint discs, (III) applying $\mathbf{X}$.

Finite derivation type

Definition

$\mathcal{P} = \langle A | R \rangle$ has finite derivation type (FDT) if there is a **finite homotopy base** for $\Gamma = \Gamma(\mathcal{P})$. A monoid M has FDT if it may be defined by a presentation with FDT.

Theorem (Squier (1994))

- *The property FDT is independent of choice of finite presentation.*
- *Let M be a finitely presented monoid. If M has a presentation by a finite complete rewriting system then M has FDT.*

Finite derivation type

some history

- **Squier (1994):** defines FDT and gives an example of a monoid with the following properties:
  - finitely presented with soluble word problem but
  - does **not** have FDT
    - hence has no presentation through a finite complete rewriting system.
- **Kobayashi (2000):** One-relator monoids have FDT
- Connections with diagram groups (which are fundamental groups of Squier complexes of monoid presentations)
  - **Kilibarda (1997)**
  - **Guba & Sapir (1997 AMS memoir)**,

Reducing semigroup theory to group theory

- $\mathcal{P}$ - property of monoids we are interested in

Idea

- Relate the problem of understanding the property for monoids with the problem of understanding the property for groups.
- One approach: via the maximal subgroups of the monoid.

Monoids and their subgroups

- M - monoid
- Green's relations $\mathcal{R}$, $\mathcal{L}$, and $\mathcal{H}$

$$x\mathcal{R}y \Leftrightarrow xM = yM, \;\; x\mathcal{L}y \Leftrightarrow Mx = My, \;\; \mathcal{H} = \mathcal{R} \cap \mathcal{L}.$$

- H = an $\mathcal{H}$-class. If H contains an idempotent e then H is a group with identity e.
  - These are precisely the maximal subgroups of M.

General question: How do the properties of M relate to those of the maximal subgroups of M?

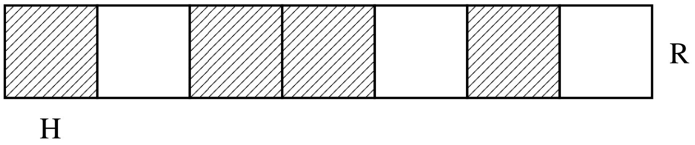
Presentations for subgroups of monoids

Theorem (Ruskuc (1999))

Let M be a monoid and let H be a maximal subgroup of M. If the $\mathcal{R}$-class of H contains only finitely many $\mathcal{H}$-classes then:

- *M finitely generated $\Rightarrow$ H finitely generated;*
- *M finitely presented $\Rightarrow$ H finitely presented.*

- **Steinberg (2003):** gave a quick topological proof in the special case of inverse semigroups
- If the finiteness assumption on the $\mathcal{R}$-class is removed then the result no longer holds.



Finite derivation type for subgroups of monoids

(joint work with A. Malheiro)

Theorem ((RG, Malheiro (2008)))

Let M be a monoid and let H be a maximal subgroup of M. If the $\mathcal{R}$-class of H contains only finitely many $\mathcal{H}$-classes then:

- *M has FDT $\Rightarrow$ H has FDT.*

Notes on proof. Given a homotopy base $\mathbf{X}$ for M we show how to construct a homotopy base $\mathbf{Y}$ for H. Finiteness is preserved when the $\mathcal{R}$-class has only finitely many $\mathcal{H}$-classes.

Regular monoids

- A semigroup is regular if every $\mathcal{R}$-class (equivalently every $\mathcal{L}$-class) contains an idempotent.

Theorem (RG, Malheiro (2008))

Let M be a regular monoid with finitely many left and right ideals. Then M has finite derivation type if and only if every maximal subgroup of M has finite derivation type.

Notes on proof. We show in general how to construct a homotopy base for M from homotopy bases of the maximal subgroups.

- **Ruskuc (1999):** Proved the corresponding result for finite generation and presentability.
- **Golubov (1975):** Showed corresponding result holds for residual finiteness.

Complete rewriting systems

Theorem (RG, Malheiro (in preparation))

Let M be a regular monoid with finitely many left and right ideals. If every maximal subgroup of M has a presentation by a finite complete rewriting system then so does M.

- The converse is still open.
- This relates to the following open problem from group theory:

  Question. Is the property of having a finite complete rewriting system preserved when taking finite index subgroups?

The finiteness condition FP_n

- **Wall (1965):** introduced a (geometric) finiteness condition for groups called $\mathcal{F}_n$:
  - $\mathcal{F}_1 \equiv$ finite generation
  - $\mathcal{F}_2 \equiv$ finite presentability
- Issue: $\mathcal{F}_n$ not very tractable in terms of using algebraic machinery
- **Bieri (1976):** introduced FP_n for groups.

Definition (in short!)

A monoid M is of type left-FP_n if $\mathbb{Z}$ has a free resolution as a trivial left $\mathbb{Z}M$-module that is finite through dimension n.

- **Kobayashi (1990):** If a monoid M is presented by a finite complete rewriting system then M is of type FP_n for all $n \in \mathbb{N}$.

$\mathbb{Z}M$-modules

- M - monoid
- $\mathbb{Z}M$ - the integral monoid ring over $\mathbb{Z}$:

  $\mathbb{Z}M = \{\sum n_u u : u \in M, n_u \in \mathbb{Z} \text{ and } n_u = 0 \text{ for all but finitely many } u\}$

  e.g. $4m_1 - 2m_2 + 3m_3 \in \mathbb{Z}M$
- Addition: $(\sum n_u u) + (\sum p_u u) = \sum (n_u + p_u) u$
- Multiplication: $(\sum n_u u)(\sum p_u u) = \sum q_u u$, where $q_u = \sum_{vw=u} n_v p_w$.
- $(\mathbb{Z}M, +)$ - a free abelian group, $(\mathbb{Z}M, +, \cdot)$ - ring
- $\mathbb{Z}M$ is a left $\mathbb{Z}M$-module where the action is the above multiplication
- Free left $\mathbb{Z}M$-module of rank $r \in \mathbb{N}$:

$$\underbrace{\mathbb{Z}M \oplus \mathbb{Z}M \oplus \cdots \oplus \mathbb{Z}M}_{r}$$

  with the natural action of $\mathbb{Z}M$ on the left.

The property FP_n

Definition

A monoid M is of type left-FP_n if there is a sequence:

$$F_n \xrightarrow{\partial_n} F_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} \mathbb{Z} \to 0$$

such that for all i we have:

- F_i is a finitely generated free left $\mathbb{Z}M$-module
- ∂_i is a homomorphism
- the sequence is exact, i.e.
  - $\partial_i(F_i) = \ker(\partial_{i-1})$, and
  - $\partial_0(F_0) = \mathbb{Z}$.

The monoid is of type left-FP_∞ if it satisfies left-FP_n for all $n \in \mathbb{N}$.

- There is an obvious dual notion of partial free resolution of right $\mathbb{Z}M$-modules, and corresponding property of right-FP_n.

FP_n in group theory

- For groups (and more generally inverse semigroups)

$$\text{left-FP}_n \equiv \text{right-FP}_n$$

- $\mathrm{FP}_1 \equiv$ finite generation
- Problem of whether

$$\mathrm{FP}_2 \equiv \text{ finite presentability ?}$$

  was open for 20 years.
- **Bestvina & Brady (1997):** answered the question in the negative

FP_n for monoids

and the relationship to FDT

- **Cohen (1992):** example of a monoid that is left-FP_∞ but not even right-FP_1!
- $\mathrm{FP}_1 \not\equiv$ finite generation & $\mathrm{FP}_2 \not\equiv$ finite presentability
- **Kobayashi (1990):**

M presented by a finite complete rewriting system	$\Rightarrow$	M is of type (left and right)-FP_∞

- **Cremanns & Otto (1994) / Lafont (1995) / Pride (1995):** For finitely presented monoids

$$\mathrm{FDT} \Rightarrow \mathrm{FP}_3.$$

- **Cremanns & Otto (1996):** for finitely presented groups

$$\mathrm{FDT} \equiv \mathrm{FP}_3.$$

A corollary about FP_3

Corollary (RG, Malheiro (2008))

Let M be a finitely presented regular monoid with finitely many left and right ideals. If every maximal subgroup of M is of type FP_3 then M is of type (left and right)-FP_3.

Proof. Every maximal subgroup $\mathrm{FP}_3 \Rightarrow$ every maximal subgroup FDT (Cremanns and Otto (1996)) $\Rightarrow M$ has FDT $\Rightarrow M$ satisfies FP_3 (Cremanns and Otto (1994)). □

This leads naturally to the following questions:

- Can the finitely presented hypothesis be lifted?
- Does the converse hold?
- What about FP_n for other values of n?

Understanding FP_1

Kobayashi's criterion

- M - monoid, $A \subseteq M$ (may not be a generating set)
- $\Gamma_r(M, A)$ - right Cayley graph of M with respect to A
- Vertex set: M
- Edge set: $x \xrightarrow{a} y$ iff $xa = y$.

Theorem (Kobayashi (2007))

Let M be a monoid. Then M is of type left-FP_1 if and only if

- *there is a finite subset A of M such that $\Gamma_r(M, A)$ is connected*

FP_n for monoids with zero

Corollary (Kobayashi (2007))

If a monoid M has a zero element z then M is of type left-FP_1.

Proof. Consider $\Gamma_r(M, A)$ where $A = \{z\}$.

Proposition (Kobayashi (preprint))

If a monoid M has a zero element z then M is if type left-FP_∞

Example

G - any group, $M = G^0$ - adjoin a zero ($0g = g0 = 00 = 0$).

- Maximal subgroups of M are: $H_1 = G$, and $H_0 = \{0\}$.
- Kobayashi $\Rightarrow$ M is left-FP_∞.
- G can have any properties we like
  - e.g. can choose G not to be of type FP_n for any given n.

Conclusion: The converse of our FP_3 result does not hold.

Clifford monoids

(joint work with S. J. Pride)

- FP_n holding in a monoid relates closely to FP_n holding in the ideals of that monoid.

Definition

Clifford monoid - a regular monoid whose idempotents are central

Theorem (RG, Pride (in preparation))

A Clifford monoid is of type left-FP_n *if and only if it has a minimal ideal* G *(which is necessarily a group) and* G *is of type left-*FP_n*.*

Completely simple semigroups

Definition

A semigroup is called simple if it has no proper ideals.

Theorem (RG, Pride (in preparation))

Let S be a simple semigroup with finitely many left and right ideals, let G be a maximal subgroup of S, and let M denote the monoid S^1. Then M is of type left-FP_n if and only if G is of type left-FP_n.

Combining the two results

- For FP_1 we have:

Theorem (RG, Pride (in preparation))

Let S be a monoid with a minimal ideal J such that J is a completely simple semigroup with finitely many left and right ideals. Let G be a maximal subgroup of J. Then S is of type left-FP_1 if and only if G is of type left-FP_1.

Corollary

Let S be a monoid with finitely many left and right ideals. Let G be a maximal subgroup of the unique minimal ideal of S. Then S is of type left-FP_1 if and only if G is of type left-FP_1.

- We have a partial proof of these results for left-FP_n in general.
- In particular we have a proof for left-FP_n when J is a group.

Future work

- Finite derivation type
  - extend results to non-regular monoids with Schützenberger groups in place of maximal subgroups
  - subsemigroups of monoids in general, Rees index, Green index.
- FP_n
  - Consider other related properties like:
    bi-FP_n, FHT, FDT_2, FHT_2, HFDT_n, etc.
  - Try to develop a better understanding of FP_n for monoids without a minimal ideal, or in the case that the minimal ideal is not completely simple with finitely many left and right ideals (e.g. B–R extensions).