Semigroups with finitely generated universal left congruence

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The aim of today

Consider two finitary conditions for semigroups and monoids:

1. ω_S^ℓ is finitely generated as a left congruence;
2. the stronger condition of S **being pseudo-finite.**

The aim of today

The aim of this work is to

1. explain the relationships between the conditions that ω_S^ℓ is finitely generated, or S is pseudo-finite, and a number of other notions such as ancestry, connected right Cayley graphs, type left-FP_1 and right unitary generation by a subset.
2. Consider closure properties under standard constructions, including morphic image, direct product, semi-direct product, free product and 0-direct union.
3. Classify certain important classes of semigroups, such as strong semilattices of semigroups, inverse semigroups, Rees matrix semigroups (over semigroups) and completely regular semigroups.

The aim of today

A question posed by Dales and White:

Is every pseudo-finite monoid isomorphic to a direct product of a monoid with zero by a finite monoid?

concerning the nature of pseudo-finite monoids, that led to this work.

Some basic properties

Let S be a semigroup.

For any $X \subseteq S$, $X^2 = X \times X$ and $\overline{X} = \{(1, x) : x \in X\}$ when S is a monoid with identity 1.

For any $A \subseteq S^2$, the left congruence on S generated by A is denoted by either ρ_A or $\langle A \rangle$.

The universal left congruence relation on S is denoted by ω_S^ℓ or simply ω^ℓ when S is not named.

Some basic properties

Lemma Let $A \subseteq S^2$. Then, for any $a, b \in S$, $a \ \rho_A \ b$ if and only if either $a = b$ or there exists a sequence

$$a = t_1 c_1, t_1 d_1 = t_2 c_2, \cdots, t_n d_n = b$$

where $t_i \in S^1$ and $(c_i, d_i) \in A \cup A^{-1}$ for all $1 \leq i \leq n$.

The sequence in the above lemma is referred to as an A-**sequence of length** n; if $n = 0$, we interpret this sequence as being $a = b$.

Notice that, for a semigroup S with ω_S^ℓ being finitely generated, we have

(1) $\omega_S^\ell = \langle X^2 \rangle$ for some finite subset $X \subseteq S$;

(2) when S is a monoid, $\omega_S^\ell = \langle \overline{X} \rangle$ where $\overline{X} = \{(1, x) : x \in X\}$ for some finite $X \subseteq S \setminus \{1\}$.

Some basic properties

Definition We say that a semigroup S is **pseudo-finite** if
(1) $\omega_S^\ell = \langle X^2 \rangle$ for some finite $X \subseteq S$;
(2) there exists $n \in \mathbb{N}$ such that for any $a, b \in S$, there is an X^2-sequence from a to b of length at most n.

Notice that the property of ω_S^ℓ being generated by a finite set, or of S being pseudo-finite is independent of the given set of generators.

It is well known and easy to see that

(1) any monoid with zero is pseudo-finite;

(2) if G is a group, then ω_G^ℓ is finitely generated if and only if G is a finitely generated group.

For arbitrary monoids and semigroups, the situation is much more complex.

Alternative conditions for ω^ℓ to be finitely generated

Theorem Let M be a monoid with a finite subset X. The following conditions are equivalent:

1. ω_M^ℓ is finitely generated by $\overline{X}$;
2. the trivial M-act Θ_M is isomorphic to $M/\rho_{\overline{X}}$ and so is finitely presented;
3. each element of M has an ancestry with respect to X;
4. the right Cayley graph $\Gamma^r(M, X)$ of M with respect to X is connected;
5. M is right unitarily generated by X.

Further, M is of type left-FP_1 if and only if any (all) of these conditions hold.

Analogous omnibus results also hold for semigroups, and for pseudo-finite monoids and semigroups.

Alternative conditions for ω^ℓ to be finitely generated

Some remarks

(1) In semigroup case, Condition 2 is stated as follows: The trivial S-act Θ_S is isomorphic to a quasi-free S-act $A = \bigcup_{i\in I} S_i$ where I is finite, factored by a finitely generated congruence.

(2) The notion of **ancestry** in a monoid was introduced by White:

Definition We say that an element $a \in M$ has an **ancestry of length** n with respect to X if there exists a finite sequence $(z_i)_{i=1}^n$ of length n in M such that $z_1 = a$, $z_n = 1$ and for each $1 < i \leq n$ there exists $x \in X$ such that either $z_i x = z_{i-1}$ or $z_i = z_{i-1}x$.

In the case of M being pseudo-finite, Condition 3 is stated as follows: every element of M has an ancestry of bounded length with respect to X.

Alternative conditions for ω^ℓ to be finitely generated

(3) In the semigroup case, Condition 4 becomes: $\Gamma_u^r(S^1, X)$ of S^1 is connected, and there is a finite set $U \subseteq S$ such that for every $a \in S$ we have $a \leq_{\mathcal{L}} u$ for some $u \in U$.

In the case of M being pseudo-finite, Condition 4 becomes: there is $n \in \mathbb{N}$ such that any two distinct vertices of $\Gamma_u^r(M, X)$ are joined by a path of length no greater than n.

(4) To obtain an analogue for Condition 5 in the semigroup case, we would require a further concept of the number of steps involved in the generation of M.

Alternative conditions for ω^ℓ to be finitely generated

(5) The property of being of type left-FP_1 does not apply to semigroups!

Definition Let M be a monoid and $\mathbb{Z}M$ be the monoid ring of M over the integers $\mathbb{Z}$. For $n \geq 0$, M is of *type left-FP_n* if there is a resolution

$$A_n \to A_{n-1} \to \cdots \to A_1 \to A_0 \to \mathbb{Z} \to 0$$

of $\mathbb{Z}$, regarded as a left $\mathbb{Z}M$-module with trivial action, such that $A_0, A_1, \cdots, A_n$ are finitely generated left $\mathbb{Z}M$-modules.

In **[Gray and Pride, 2011]** a semigroup is said to be of type left-FP_1 if S^1 is of type left-FP_1.

For semigroups, the property of ω_S^ℓ being finitely generated and that of $\omega_{S^1}^\ell$ being finitely generated differ considerably.

Closure properties under standard constructions

Standard constructions considered here include morphisms, direct products, semidirect products, free products and 0-direct unions.

Two kinds of questions:

(1) whether the class of semigroups with ω^{ℓ} being finitely generated is closed under a particular construction;

(2) whether the fact that ω^{ℓ} is finitely generated passes down to components of the construction.

Closure properties under standard constructions

The following result is known in the case of a retract of a monoid **[Pride, 2016]**.

Proposition Let S be a semigroup and let T be a morphic image of S. If ω_S^ℓ is finitely generated (S is pseudo-finite), then ω_T^ℓ is finitely generated (T is pseudo-finite).

Closure properties under standard constructions

[Guba and Pride, 1998]

A direct product of a pair of monoids M and N is shown (in the context of being of type left-FP_1) to have the property that $\omega^{\ell}_{M\times N}$ is finitely generated if and only if both ω^{ℓ}_{M} and ω^{ℓ}_{N} are finitely generated.

Proposition Let S and T be semigroups. If $\omega^{\ell}_{S\times T}$ is finitely generated ($S \times T$ is pseudo-finite) then both ω^{ℓ}_{S} and ω^{ℓ}_{T} are finitely generated (pseudo-finite).

Notice that the converse does not necessarily hold for semigroups.

Closure properties under standard constructions

Proposition Let S and T be monoids such that $\omega_S^\ell = \langle U^2 \rangle$ and $\omega_T^\ell = \langle V^2 \rangle$ for some finite subsets $U \subseteq S$ and $V \subseteq T$. Suppose T acts monoidally on S by morphisms. Then $\omega_{S \rtimes T}^\ell$ is finitely generated. Moreover, if S and T are pseudo-finite, then so is $S \rtimes T$.

Notice that,

(1) the action of T on S being monoidal is necessary;

(2) if $\omega_{S \rtimes T}^\ell$ is finitely generated ($S \rtimes T$ is pseudo-finite), then so is T. But it is not true for S, in general.

Closure properties under standard constructions

Proposition Let S and T be semigroups. Then ω^{ℓ}_{S*T} is finitely generated if and only if ω^{ℓ}_{S} and ω^{ℓ}_{T} are finitely generated. The semigroup $S * T$ is never pseudo-finite.

For the case of the monoid free product of two monoids $M *^m N$, **[Kobayashi, 2010]** showed that if M and N are finitely presented, then $\omega^{\ell}_{M*^m N}$ is finitely generated if ω^{ℓ}_{M} and ω^{ℓ}_{N} are.

Corollary Let M and N be monoids. Then $\omega^{\ell}_{M*^m N}$ is finitely generated if and only if ω^{ℓ}_{M} and ω^{ℓ}_{N} are finitely generated. The monoid $M *^m N$ is never pseudo-finite unless one of M, N is pseudo-finite and the other is trivial.

Proposition Let S and T be semigroups with zero and let $P = S \cup T$ be the 0-direct union of S and T. Then ω^{ℓ}_{P} is finitely generated if and only if both ω^{ℓ}_{S} and ω^{ℓ}_{T} are finitely generated. Moreover, P is pseudo-finite if and only if both S and T are pseudo-finite.

Inverse semigroups

Theorem Let S be an inverse semigroup and $E(S)$ be the set of idempotents of S. Then the following statements are equivalent:

1. ω_S^ℓ is finitely generated;
2. (i) there is a finite set $U \subseteq E(S)$ such that for every $e \in E(S)$ we have $e \leq u$ for some $u \in U$; and
(ii) there is a finitely generated inverse subsemigroup W of S such that for all $a \in S$ and $e \in E(W)$, there exists $w \in W$ with $aw = ew^{-1}w$;
3. (i) there is a finite set $U \subseteq E(S)$ such that for every $e \in E(S)$ we have $e \leq u$ for some $u \in U$; and
(iii) there is a finitely generated inverse subsemigroup W of S such that for all $a \in S$ there exists $w \in W$ with $aw \in E(W)$.

Theorem Suppose that S is an inverse semigroup with semilattice of idempotents $E(S)$. Then S is pseudo-finite if and only if there is a finite set $U \subseteq E(S)$ such that for every $f \in E(S)$ we have $f \leq u$ for some $u \in U$; $E(S)$ has a least element e, and the group $\mathcal{H}$-class H_e is finite.

Specialize to inverse monoids.

Corollary Let E be a semilattice. Then the following statements are equivalent:

1. ω_E^ℓ is finitely generated;
2. E is pseudo-finite;
3. E has a least element and there is a finite set $U \subseteq E$ such that for every $e \in E$ we have $e \leq u$ for some $u \in U$.

Corollaries [Gray and Pride, 2011]
Let M be an inverse monoid with a least idempotent e. Then ω_M^ℓ is finitely generated if and only if H_e is finitely generated.

Let E be a semilattice with identity 1. Then ω_E^ℓ is finitely generated if and only if E is pseudo-finite if and only if E has a least element.

Notice that the existence of a least idempotent is not necessary for ω^ℓ to be finitely generated.

Rees matrix semigroups

There are four cases for us to consider that arise from the existence or otherwise of an identity, and the existence or otherwise of a zero:

1. $T = \mathcal{M}^0[S; I, \Lambda; P]^1$;
2. $T = \mathcal{M}[S; I, \Lambda; P]$;
3. $T = \mathcal{M}^0[S; I, \Lambda; P]$;
4. $T = \mathcal{M}[S; I, \Lambda; P]^1$.

Rees matrix semigroups

Theorem Let $T = \mathcal{M}[S; I, \Lambda; P]^1$ be a Rees matrix semigroup over a semigroup S, with identity adjoined. Then ω_T^ℓ is finitely generated if and only if the following conditions hold:

1. I is finite;
2. there is a finite set $V \subseteq S$ and a finite subset Q of entries of $P = (p_{\lambda,i})$ such that any element a of S is ρ_U-related to an element of V via the left congruence ρ_U defined on S^1, where U is defined as

$$\{(p_{\nu i}a, p_{\nu j}b) : \nu \in \Lambda, i, j \in I, a, b \in V\} \cup \{(1, pv), (pv, 1) : p \in Q, v \in V\}$$

via a U-sequence of the form

$$a = t_1c_1, t_1d_1 = t_2c_2, \ldots, t_kd_k = v,$$

where $t_i \in S$ and $(d_j, c_{j+1}) \neq (1, 1)$ for any $1 \leq j < k$.

Rees matrix semigroups

Specialise to the case where S is a group and T is a rectangular band.

In the context of being pseudo-finite, we merely need to impose a bound on the length of the sequences to achieve the criterion for being pseudo-finite.

Notice that, in the case where S is a group, namely, completely simple semigroups of type left-FP_1 were considered in **[Gray and Pride, 2011]**, but the convention there is that one considers the property for the corresponding monoid obtained by adjoining an identity.

Strong semilattices of semigroups and Bruck-Reilly extensions

Strong semilattices of monoids which are of type left-FP_n, and hence of type left-FP_1, have been classified in **[Gray and Pride, 2011]**.

Generalise this result to strong semilattices of semigroups, with and without an identity adjoined, and to examine the property of being pseudo-finite.

Theorem Let $S = [\mathcal{Y}; S_\alpha; \phi_{\alpha,\beta}]$ be a strong semilattice of semigroups. Then ω_S^ℓ is finitely generated (S is pseudo-finite) if and only if

1. there exists a finite subset X of S such that for every $a \in S$ we have some $x \in X$ with $a \leq_{\mathcal{L}} x$;
2. $\mathcal{Y}$ has a least element 0;
3. $\omega_{S_0}^\ell$ is finitely generated (S_0 is pseudo-finite).

Strong semilattices of semigroups and Bruck-Reilly extensions

Theorem Let S be a monoid with ω_S^ℓ being finitely generated and let $T = BR(S, \theta)$ be the Bruck-Reilly extension of S determined by θ. Then ω_T^ℓ is finitely generated, but T is not pseudo-finite.

Notice that, the property of ω^ℓ being finitely generated for $\mathrm{BR}(S, \theta)$ does not necessarily transfer to S.

Using a strong semilattice of groups we are able to give a counterexample to the conjecture of Dales and White: Every pseudo-finite monoid is isomorphic to a direct product of a monoid with zero by a finite monoid.

Semigroups with a minimum ideal and completely regular semigroups

Notice that, for Clifford semigroups and normal bands with ω_S^ℓ being finitely generated, the underlying semilattice $\mathcal{Y}$ has a least element 0 and S_0 is the minimum ideal with $\omega_{S_0}^\ell$ being finitely generated.

Characterise a semigroup with a completely simple minimum ideal to have ω^ℓ finitely generated.

Semigroups with a minimum ideal and completely regular semigroups

Theorem Let S be a semigroup with a minimum ideal S_0 that is completely simple. Then ω_S^ℓ is finitely generated if and only if the following hold:

1. there exists a finite subset X of S such that for every $a \in S$ we have some $x \in X$ with $a \leq_{\mathcal{L}} x$;
2. if G is a maximal subgroup of S_0, then $G = \langle F \cup V \rangle$ where V is finite and $F = \langle E(C_0) \rangle \cap G$ where C_0 is the union of finitely many $\mathcal{R}$-classes of S_0;
3. there exists an $\mathcal{L}$-class L of S_0 and a finite subset $W \subseteq L$ such that every idempotent in L is ρ_{W^2}-related to an element of W via the left congruence ρ_{W^2} defined on S.

Semigroups with a minimum ideal and completely regular semigroups

Specialise to the case where S_0 with finitely many $\mathcal{R}$-classes-this can be regarded as an extension of the result in **[Gray and Pride, 2011]** from monoids to semigroups.

Specialise to completely regular semigroups-notice that we make no restriction here on the number of right ideals of the minimum ideal, unlike the result in **[Gray and Pride, 2011]**, and bands.

Notice that in current examples of completely regular semigroups with finitely generated universal, including completely simple semigroups and strong semilattices of groups or rectangular bands, the minimum ideals have finitely many $\mathcal{R}$-classes.

There exists a left regular band monoid with finitely generated universal relation such that the minimum ideal has infinitely many $\mathcal{R}$-classes.

Thanks for listening!

Any Questions?