The global dimension of the algebra of the monoid of all partial functions on an *n*-set

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Monoid algebras

- M - finite monoid.
- $\mathbb{C}M$ - monoid algebra.

$$\mathbb{C}M = \{\sum \alpha_i m_i \mid \alpha_i \in \mathbb{C} \quad m_i \in M\}$$

- $\mathbb{C}M$ is usually not a semisimple algebra.

Question

Given an interesting monoid M, try to find properties\invariants of $\mathbb{C}M$

Monoid algebras

- Interesting choices of M:
  - Transformation monoid: $\mathcal{T}_n$, $\mathcal{PT}_n$, $\mathcal{IS}_n$, order-related monoids, etc.
  - Classes: $\mathcal{J}$-trivial monoids, $\mathcal{R}$-trivial monoids, left regular bands, **DO** monoids.
- Invariants:
  - Character table, Jacobson radical, Projective\Injective\Simple modules, Cartan matrix, Quiver, Quiver presentation, Global dimension.

Goal

For our talk:

- $M = \mathcal{PT}_n$. The monoid of all partial functions on $\{1, \ldots, n\}$. 
- Invariant = The global dimension.

Goal

Find the global dimension of $\mathbb{C}\mathcal{PT}_n$.

Known results on global dimension of monoid algebras

- Steinberg (2016):$\mathrm{gl.Dim}(\mathbb{C}\mathcal{T}_n) = n - 1$.
- Margolis, Saliola, Steinberg (2015): Certain results on the global dimension of (algebras of) left regular bands.

Outline

- Preliminaries on Rep Theory of $\mathcal{PT}_n$.
- Cartan Matrix
- Quiver
- Global dimension

$\mathcal{PT}_n$

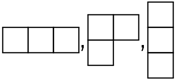
- The monoid $\mathcal{PT}_n$.
  - Regular.
  - The $\mathcal{J}$ order is linear.
  - The maximal subgroups are S_k where $0 \leq k \leq n$.
  - The structure matrix ("Rees sandwich matrix") of J_k is left invertible over $\mathbb{C}S_k$.

Munn-Ponizovsky

Theorem (Munn-Ponizovsky)

Let M be a finite monoid. There is a one-to-one correspondence between simple modules of M and simple modules of its maximal subgroups.

- The maximal subgroups of $\mathcal{PT}_n$ are S_k for $0 \leq k \leq n$
- Irreducible representations of S_n can be parameterized by partitions $\alpha \vdash n$, or equivalently, by Young diagrams with n-boxes:

- Irreducible representations of $\mathcal{PT}_n$ can be parameterized by partitions $\alpha \vdash k$ for $0 \leq k \leq n$, or equivalently, by Young diagrams.

Rep. Theory of $\mathcal{PT}_n$ using a Category

Definition

Let D be a finite category. The category algebra $\mathbb{C}D$ consists of linear combination of morphisms

$$\{\sum \alpha_i m_i \mid \alpha_i \in \mathbb{C} \quad m_i \in MC^1\}$$

with multiplication being linear extension of

$$m_1 \cdot m_2 = \begin{cases} m_1 m_2 & \text{if defined} \\ 0 & \text{otherwise} \end{cases}$$

Rep. Theory of $\mathcal{PT}_n$ using E_n

Definition

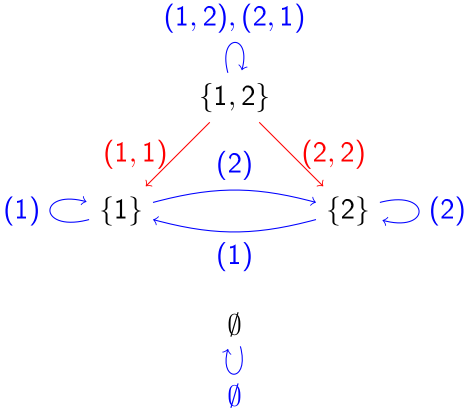
Let E_n be the category whose objects are the subsets of $\{1 \ldots n\}$, and whose morphisms are all the total onto functions between subsets. (There is a one-to-one correspondence between morphisms and elements of $\mathcal{PT}_n$).

Remark

For every object X, its endomorphisms form the group $S_{|X|}$.

Rep. Theory of $\mathcal{PT}_n$ using E_n

E_2:



Rep. Theory of $\mathcal{PT}_n$ using E_n

Theorem (IS 2016)

$\mathbb{C}\mathcal{PT}_n \cong \mathbb{C}E_n$.

Remark

Similar result holds for many other finite semigroups.

- Lattices (Solomon 1967),
- Inverse semigroups (Steinberg 2006),
- Ample semigroups (Guo, Chen 2012)
- Ehresmann+left\right restriction (IS 2017),
- P-Ehresmann+ left\right P-restriction (Wang 2017)

Outline

- Preliminaries on Rep Theory of $\mathcal{PT}_n$
- Cartan Matrix
- Quiver
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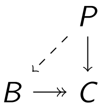
Projective modules

Definition

Let A be be a finite dimensional algebra over $\mathbb{C}$ and let $\mathrm{Hom}_A(M, -) : A - \mathbf{Mod} \to \mathbf{Ab}$ be the usual hom functor. An A-module P is called *projective* if $\mathrm{Hom}_A(P, -)$ is an exact functor.



Projective modules

- $\text{Ext}^n(P, N) = 0$ for every projective P.
- There is a one-to-one corrspondence between simple modules and indecomposable projective modules.
- Therefore: Indecomposable Projective modules of $\mathbb{C}\mathcal{PT}_n$ are also parameterized by Young diagrams $\alpha \vdash k$ for $0 \leq k \leq n$,.

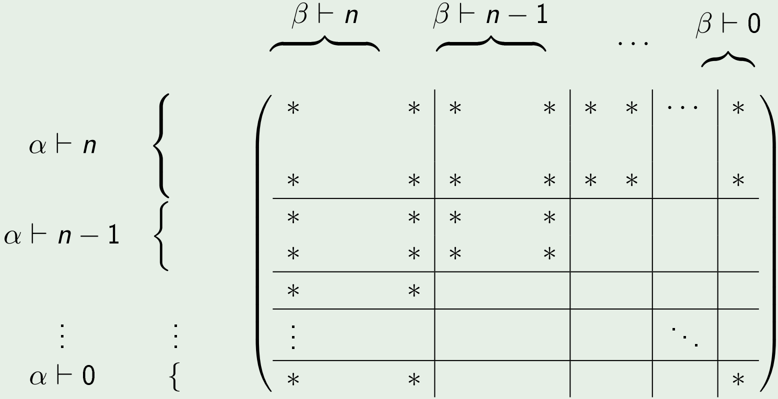
Cartan matrix - Definition

Definition

Let $S(1), \dots S(n)$ be the simple modules of A with corresponding indecomposable projective modules $P(1), \dots, P(n)$. The *cartan matrix* of A is an $n \times n$ matrix whose (a, b) entry is the number of times that $S(a)$ appears as a Jordan-Hölder factor of $P(b)$.

Cartan matrix of $\mathcal{PT}_n$

For $\mathcal{PT}_n$ the simples\projectives are indexed by Young diagrams $\alpha \vdash k$ and $\beta \vdash r$. How many times $S(\alpha)$ appears as a Jordan-Hölder factor of $P(\beta)$?



Cartan matrix of $\mathcal{PT}_n$

Proposition (Putcha 1995)

The Cartan matrix of $\mathcal{PT}_n$ is block lower-unitriangular.

$$
\begin{array}{c c}
 & \begin{array}{cccc} \overbrace{\qquad\qquad}^{\beta \vdash n} & \overbrace{\qquad\qquad}^{\beta \vdash n-1} & \cdots & \overbrace{\quad}^{\beta \vdash 0} \end{array} \\
\begin{array}{l} \alpha \vdash n \; \left\{ \vphantom{\begin{matrix} 0 \\ \\ 0 \end{matrix}} \right. \\ \alpha \vdash n-1 \; \left\{ \vphantom{\begin{matrix} * \\ * \end{matrix}} \right. \\ \\ \vdots \qquad \vdots \\ \alpha \vdash 0 \; \{ \end{array} &
\left(\begin{array}{cc|cc|cc|c|c}
 & & 0 & 0 & 0 & 0 & \cdots & 0 \\
 & I & & & & & & \\
 & & 0 & 0 & 0 & 0 & & 0 \\ \hline
* & * & \ddots & & 0 & 0 & & 0 \\
* & * & & & 0 & 0 & & 0 \\ \hline
* & * & & & \ddots & & & \\ \hline
\vdots & & & & & & \ddots & \\ \hline
* & * & & & & & & I
\end{array}\right)
\end{array}
$$

Cartan matrix of $\mathcal{PT}_n$

Question

What about the other elements of the matrix?

$$
\begin{array}{c c}
 & \overbrace{\qquad\quad}^{\beta \vdash n} \quad \overbrace{\qquad\quad}^{\beta \vdash n-1} \quad \cdots \quad \overbrace{\quad}^{\beta \vdash 0} \\
\begin{array}{l}
\alpha \vdash n \quad \left\{ \vphantom{\begin{array}{c} 0 \\ 0 \\ 0 \end{array}} \right. \\
\alpha \vdash n-1 \quad \left\{ \vphantom{\begin{array}{c} 0 \\ 0 \end{array}} \right. \\
\vdots \qquad\quad \vdots \\
\alpha \vdash 0 \quad \{
\end{array}
&
\left(
\begin{array}{cc|cc|cc|c|c}
 & & 0 & 0 & 0 & 0 & \cdots & 0 \\
\multicolumn{2}{c|}{I} & & & & & & \\
 & & 0 & 0 & 0 & 0 & & 0 \\
\hline
* & * & \ddots & & 0 & 0 & & 0 \\
* & * & & & 0 & 0 & & 0 \\
\hline
* & * & & & \ddots & & & \\
\hline
\vdots & & & & & & \ddots & \\
\hline
* & * & & & & & & I
\end{array}
\right)
\end{array}
$$

Cartan matrix

Define $E(r,k)$ to be the set of all onto **total** functions from $\{1,\ldots,r\}$ to $\{1,\ldots,k\}$. This is an $S_k \times S_r$ module via action $(\pi,\tau)*f = \pi f \tau^{-1}$.
Given a partition $\alpha \vdash n$, denote by S^α the Specht module (=irreducible S_n-representation) corresponding to α.
The irreducible representations of $S_k \times S_r$ are $\{S^\alpha \otimes S^\beta \mid \alpha \vdash k, \quad \beta \vdash r\}$.

Proposition (IS)

The number of times that $S(\alpha)$ appears as a J-H factor in $P(\beta)$ is the number of times that $S^\alpha \otimes S^\beta$ appears as an irreducible constituent in $E(r,k)$.

Remark

Similar to other descriptions of the Cartan matrix in the literature.

Rep. Theory of S_n

- Let G be a group and $H \leq G$ a subgroup. Let V (U) be an irreducible G-module (resp. H-module). We denote by $\operatorname{Res}_H^G V$, $\operatorname{Ind}_H^G U$ the usual induction and restriction functors.
- If $G = S_n$ and $H = S_{n-1}$ then $\operatorname{Res}_{S_{n-1}}^{S_n} V$ ($\operatorname{Ind}_{S_{n-1}}^{S_n} U$) is obtained by removing (resp. adding) boxes from the corrsponding diagram ("Classical" branching rules).
- If $G = S_n$ and $H = S_k \times S_{n-k}$ then $\operatorname{Ind}_{S_k \times S_{n-k}}^{S_n} U$ is described by the Littlewood-Richardson branching rule.

Proposition (IS 2016)

Explicit description of the block diagonal below the main diagonal.

$$
\begin{array}{r l}
 & \begin{array}{cccc} \overbrace{\qquad\qquad}^{\beta \vdash n} & \overbrace{\qquad\qquad}^{\beta \vdash n-1} & \cdots & \overbrace{\quad}^{\beta \vdash 0} \end{array} \\
\begin{array}{r} \alpha \vdash n \left\{ \begin{array}{c} \\ \\ \end{array} \right. \\ \alpha \vdash n-1 \left\{ \begin{array}{c} \\ \\ \end{array} \right. \\ \vdots \qquad \vdots \\ \alpha \vdash 0 \quad \{ \end{array}
&
\left(\begin{array}{cc|cc|cc|c|c}
 & & 0 & 0 & 0 & 0 & \cdots & 0 \\
 & I & & & & & & \\
 & & 0 & 0 & 0 & 0 & & 0 \\ \hline
\checkmark & \checkmark & \ddots & & 0 & 0 & & 0 \\
\checkmark & \checkmark & & & 0 & 0 & & 0 \\ \hline
* & * & \checkmark & & \ddots & & & \\ \hline
\vdots & & & & \checkmark & & \ddots & \\ \hline
* & * & & & & & \checkmark & I
\end{array}\right)
\end{array}
$$

Cartan matrix

Proposition (IS 2016)

Let $\alpha \vdash k$ and $\beta \vdash k+1$. The number of times that $S(\alpha)$ appears as a J-H factor in $P(\beta)$ is the number of times that S^{β} appears as an irreducible constituent in

$$\mathrm{Ind}_{S_{k-1}\times S_2}^{S_{k+1}}(\mathrm{Res}_{S_{k-1}}^{S_k} S^{\alpha} \otimes \mathrm{tr}_{S_2})$$

which is the number of ways to obtain β from α by removing one box and adding two but not in the same column.

Cartan matrix

Proposition (IS)

Explicit description of another block sub-diagonal.

$$
\begin{array}{ll}
 & \\
\alpha \vdash n & \{ \\
\alpha \vdash n-1 & \{ \\
\vdots & \vdots \\
\alpha \vdash 0 & \{
\end{array}
\quad
\overbrace{\quad}^{\beta \vdash n} \; \overbrace{\quad}^{\beta \vdash n-1} \; \cdots \; \overbrace{\quad}^{\beta \vdash 0}
$$

$$
\left(\begin{array}{cc|cc|cc|c|c}
 & & 0 & 0 & 0 & 0 & \cdots & 0 \\
\multicolumn{2}{c|}{I} & & & & & & \\
 & & 0 & 0 & 0 & 0 & & 0 \\
\hline
\checkmark & \checkmark & \ddots & & 0 & 0 & & 0 \\
\checkmark & \checkmark & & & 0 & 0 & & 0 \\
\hline
\checkmark & \checkmark & \multicolumn{2}{c|}{\checkmark} & \ddots & & & \\
\hline
\vdots & & & & \checkmark & & \ddots & \\
\hline
* & * & & & \checkmark & & \checkmark & I
\end{array}\right)
$$

Cartan matrix

Proposition (IS)

Let $\alpha \vdash k$ and $\beta \vdash k+2$. The number of times that $S(\alpha)$ appears as a J-H factor in $P(\beta)$ is the number of times that S^{β} appears as an irreducible constituent in

$$\mathrm{Ind}_{S_{k-1}\times S_3}^{S_{k+2}}(\mathrm{Res}_{S_{k-1}}^{S_k}(S^{\alpha}) \otimes \mathrm{tr}_{S_3}) \oplus \mathrm{Ind}_{S_{k-2}\times D_4}^{S_{k+2}} \overline{\mathrm{Res}_{S_{k-2}\times S_2}^{S_k} S^{\alpha}}.$$

Outline

- Preliminaries on Rep Theory of $\mathcal{PT}_n$
- Cartan Matrix
- Quiver
- Global dimension

Quiver of $\mathcal{PT}_n$

Definition

Let A be an algebra. The quiver of A is the directed graph Q defined as follows:

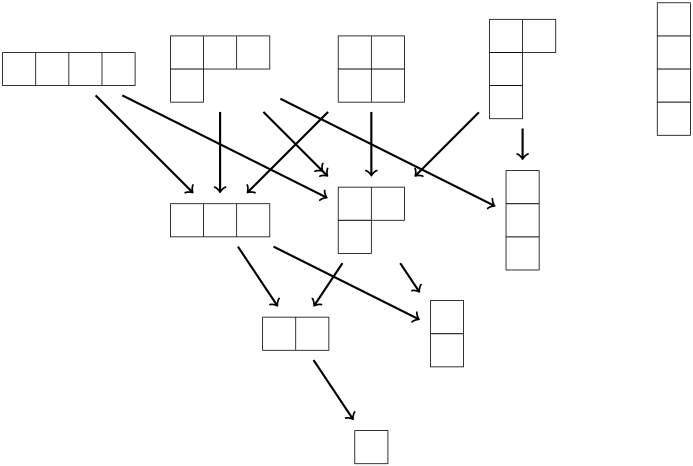
- Vertices - Simple modules.
- Edges - The number of edges between S_1 to S_2 is $\dim \operatorname{Ext}^1(S_1, S_2)$.

Theorem (IS 2016)

: Computation of the Quiver of $\mathbb{C}\mathcal{PT}_n$.

- *Vertices: Young diagrams with k-boxes.*
- *Edges: $\sharp\{\beta \to \alpha\} =$ number of ways to obtain β from α by removing one box and adding two but not in the same column.*

Quiver of $\mathbb{C}\mathcal{PT}_4$



$\emptyset$

Outline

- Preliminaries on Rep Theory of $\mathcal{PT}_n$
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Global dimension - definition

Definition

Let M be an A-module. A *projective resolution* of M is an exact sequence

$$0 \to P_n \to \ldots \to P_1 \to P_0 \to M \to 0$$

where every P_i is projective.
$n =$ length of the projective resolution.

Definition

The projective dimension of M is the minimal length of a projective resolution of M.

Example

$\mathrm{pd}(M) = 0 \iff M$ is projective.

Global dimension - definition

- $\mathrm{Ext}^n(M, -)$ - the n-th right derived functor of $\mathrm{Hom}(M, -)$.

Fact

$\mathrm{pd}(M) = \min\{m \mid \mathrm{Ext}^{m+1}(M, N) = 0 \textit{ for every } N \in A - \mathbf{Mod}\}$

Global dimension - definition

Definition

The *global dimension* of an algebra A is

$$\text{gl. Dim}(A) = \sup\{\text{pd}(M) \mid M \in \text{A} - \textbf{Mod}\}$$

Upper bound

Theorem (Nico's Theorem)

Let M be a regular monoid and let k be the longest chain in the $\mathcal{J}$-order. Then $\operatorname{gl.Dim}(\mathbb{C}M) \leq 2k$.
If all the structure matrices are left or right invertible, then $\operatorname{gl.Dim}(\mathbb{C}M) \leq k$.

- For $M = \mathbb{C}\mathcal{PT}_n$ this gives $\operatorname{gl.Dim}(\mathbb{C}\mathcal{PT}_n) \leq n$.
- It is easy to show we can ignore the $\mathcal{J}$ class of the zero function, so actually $\operatorname{gl.Dim}(\mathbb{C}\mathcal{PT}_n) \leq n - 1$.
- Equivalent: The global dimension is bounded abve by the longest path in the quiver.

Main Theorem

Theorem

$\mathrm{gl.\,Dim}(\mathbb{C}\mathcal{PT}_n) = n - 1.$

- It is enough to find a module M with $\mathrm{pd}(M) = n - 1$.
- It is enough to find modules M, N with $\mathrm{Ext}^{n-1}(M, N) \neq 0$.

The projective module of the “dual standard” partition

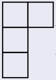
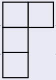
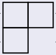
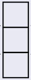
Conjecture (Walter Mazorchuk)

Consider the projective indecomposable module $P(\beta)$ for the partition $\beta = [2, 1^{n-2}]$. It contains only few J-H components.

Proposition (IS)

For $n \geq 3$, the only J-H components of $P(\beta)$ are the simples for $[2, 1^{n-2}]$, $[2, 1^{n-3}]$ and $[1^{n-1}]$. Each one with multiplicity 1.

Example ($n = 4$)

The J-H components of $P($$)$ are $S($$)$,$S($$)$ and $S($$)$.

Homological arguments for n=4

- Consider the short exact sequence

$$0 \to \operatorname{Rad} P\left(\ydiagram{2,1,1}\right) \to P\left(\ydiagram{2,1,1}\right) \to S\left(\ydiagram{2,1,1}\right) \to 0$$

- By the above we know that the J-H components of Rad $P\left(\ydiagram{2,1,1}\right)$ are $S\left(\ydiagram{2,1}\right)$ and $S\left(\ydiagram{1,1,1}\right)$.
- Other known facts:
  - $S\left(\ydiagram{1,1,1}\right)$is a projective module.
  - $\operatorname{Ext}^1\left(S\left(\ydiagram{2,1}\right), S\left(\ydiagram{1,1,1}\right)\right) = \operatorname{Ext}^1\left(S\left(\ydiagram{1,1,1}\right), S\left(\ydiagram{2,1}\right)\right) = 0$

    This implies that Rad $P\left(\ydiagram{2,1,1}\right) = S\left(\ydiagram{2,1}\right) \oplus S\left(\ydiagram{1,1,1}\right)$.

Homological arguments for n=4

- Consider the short exact sequence

$$0 \to S(\yng(2,1)) \oplus S(\yng(1,1,1)) \to P(\yng(2,1,1)) \to S(\yng(2,1,1)) \to 0$$

- By the "long exact sequence" Theorem we have that

$$\mathrm{Ext}^k(S(\yng(2,1,1)), S(\yng(1))) \cong \mathrm{Ext}^{k-1}(S(\yng(2,1)) \oplus S(\yng(1,1,1)), S(\yng(1)))$$

$$\mathrm{Ext}^{k-1}(S(\yng(2,1)) \oplus S(\yng(1,1,1)), S(\yng(1))) =$$

$$\mathrm{Ext}^{k-1}(S(\yng(2,1)), S(\yng(1))) \oplus \mathrm{Ext}^{k-1}(S(\yng(1,1,1)), S(\yng(1))) =$$

$$\mathrm{Ext}^{k-1}(S(\yng(2,1)), S(\yng(1)))$$

Homological arguments for n=4

- Hence

$$\mathrm{Ext}^k(S(\ydiagram{2,1,1}), S(\ydiagram{1})) \cong \mathrm{Ext}^{k-1}(S(\ydiagram{2,1}), S(\ydiagram{1}))$$

- This implies that

$$\mathrm{pd}(S(\ydiagram{2,1,1})) = \mathrm{pd}(S(\ydiagram{2,1})) + 1$$

- This implies that

$$\mathrm{pd}(S([2,1^{n-2}])) = \mathrm{pd}(S([2,1^{n-2}])) + 1$$

Homological arguments

- In general we prove that

$$\mathrm{pd}(S([2,1^{n-2}])) = \mathrm{pd}(S([2,1^{n-2}])) + 1$$

- This implies that

$$\mathrm{pd}(S([2,1^{n-2}])) = n-1$$

- Therefore:

$$\mathrm{gl.\,Dim}(\mathbb{C}\mathcal{PT}_n) = n-1$$

as required.

Thank you!