Orbital profile and orbit algebra of oligomorphic permutation groups

Conjectures of Cameron and Macpherson

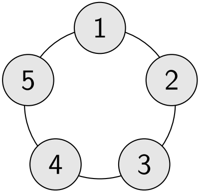
Justine Falque
joint work with Nicolas M. Thiéry

Laboratoire de Recherche en Informatique
Université Paris-Sud (Orsay)

NBSAN, June 14th of 2018

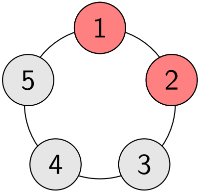
Age and profile : example on a finite group (1)

Action of the cyclic group $G = \mathcal{C}_5$ on the five pearl necklace
$\rightarrow$ induced action on *subsets* of pearls



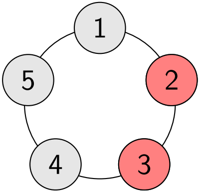
Age and profile : example on a finite group (1)

Action of the cyclic group $G = \mathcal{C}_5$ on the five pearl necklace
$\rightarrow$ induced action on *subsets* of pearls



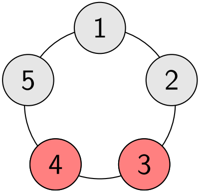
Age and profile : example on a finite group (1)

Action of the cyclic group $G = \mathcal{C}_5$ on the five pearl necklace
$\rightarrow$ induced action on *subsets* of pearls



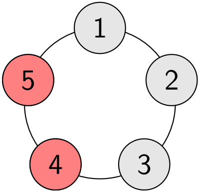
Age and profile : example on a finite group (1)

Action of the cyclic group $G = \mathcal{C}_5$ on the five pearl necklace
$\rightarrow$ induced action on *subsets* of pearls



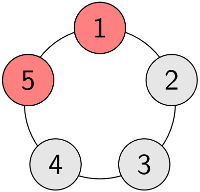
Age and profile : example on a finite group (1)

Action of the cyclic group $G = \mathcal{C}_5$ on the five pearl necklace
$\rightarrow$ induced action on *subsets* of pearls



Age and profile : example on a finite group (1)

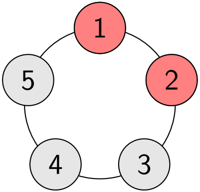
Action of the cyclic group $G = \mathcal{C}_5$ on the five pearl necklace
$\rightarrow$ induced action on *subsets* of pearls



Age and profile : example on a finite group (1)

Action of the cyclic group $G = \mathcal{C}_5$ on the five pearl necklace
$\rightarrow$ induced action on *subsets* of pearls

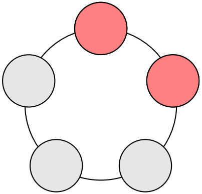
Degree of an orbit : the cardinality shared by all subsets in that orbit



Age and profile : example on a finite group (1)

Action of the cyclic group $G = \mathcal{C}_5$ on the five pearl necklace
$\rightarrow$ induced action on *subsets* of pearls

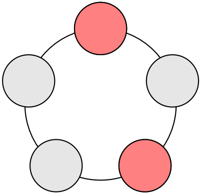
Degree of an orbit : the cardinality shared by all subsets in that orbit



Age and profile : example on a finite group (1)

Action of the cyclic group $G = \mathcal{C}_5$ on the five pearl necklace
$\rightarrow$ induced action on *subsets* of pearls

Degree of an orbit : the cardinality shared by all subsets in that orbit



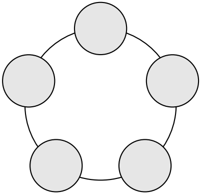
Age and profile : example on a finite group (1)

Action of the cyclic group $G = \mathcal{C}_5$ on the five pearl necklace
$\rightarrow$ induced action on *subsets* of pearls

Degree of an orbit : the cardinality shared by all subsets in that orbit

Age of G : $\mathcal{A}(G) = \sqcup_n \mathcal{A}(G)_n, \quad \mathcal{A}(G)_n = \{\text{orbits of degree } n\}$

Profile of G : $\varphi_G : n \mapsto \text{card}(\mathcal{A}(G)_n)$



Age and profile : example on a finite group (1)

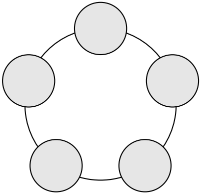
Action of the cyclic group $G = \mathcal{C}_5$ on the five pearl necklace
$\rightarrow$ induced action on *subsets* of pearls

Degree of an orbit : the cardinality shared by all subsets in that orbit

Age of G : $\mathcal{A}(G) = \sqcup_n \mathcal{A}(G)_n, \quad \mathcal{A}(G)_n = \{\text{orbits of degree } n\}$

Profile of G : $\varphi_G : n \mapsto \text{card}(\mathcal{A}(G)_n)$

$\varphi_G(0) = 1$



Age and profile : example on a finite group (1)

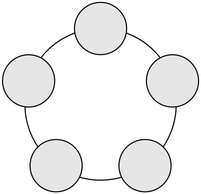
Action of the cyclic group $G = \mathcal{C}_5$ on the five pearl necklace
$\to$ induced action on *subsets* of pearls

Degree of an orbit : the cardinality shared by all subsets in that orbit

Age of G : $\mathcal{A}(G) = \sqcup_n \mathcal{A}(G)_n, \quad \mathcal{A}(G)_n = \{\text{orbits of degree } n\}$

Profile of G : $\varphi_G : n \mapsto \text{card}(\mathcal{A}(G)_n)$

$\varphi_G(0) = 1$
$\varphi_G(1)$



Age and profile : example on a finite group (1)

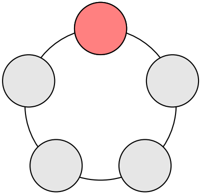
Action of the cyclic group $G = \mathcal{C}_5$ on the five pearl necklace
$\rightarrow$ induced action on *subsets* of pearls

Degree of an orbit : the cardinality shared by all subsets in that orbit

Age of G : $\mathcal{A}(G) = \sqcup_n \mathcal{A}(G)_n, \quad \mathcal{A}(G)_n = \{\text{orbits of degree } n\}$

Profile of G : $\varphi_G : n \mapsto \text{card}(\mathcal{A}(G)_n)$

$\varphi_G(0) = 1$
$\varphi_G(1)$



Age and profile : example on a finite group (1)

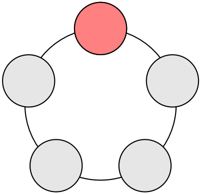
Action of the cyclic group $G = \mathcal{C}_5$ on the five pearl necklace
$\rightarrow$ induced action on *subsets* of pearls

Degree of an orbit : the cardinality shared by all subsets in that orbit

Age of G : $\mathcal{A}(G) = \sqcup_n \mathcal{A}(G)_n, \quad \mathcal{A}(G)_n = \{\text{orbits of degree } n\}$

Profile of G : $\varphi_G : n \mapsto \text{card}(\mathcal{A}(G)_n)$

$\varphi_G(0) = 1$
$\varphi_G(1) = 1$



Age and profile : example on a finite group (1)

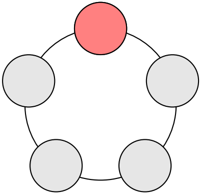
Action of the cyclic group $G = \mathcal{C}_5$ on the five pearl necklace
$\rightarrow$ induced action on *subsets* of pearls

Degree of an orbit : the cardinality shared by all subsets in that orbit

Age of G : $\mathcal{A}(G) = \sqcup_n \mathcal{A}(G)_n, \quad \mathcal{A}(G)_n = \{\text{orbits of degree } n\}$

Profile of G : $\varphi_G : n \mapsto \text{card}(\mathcal{A}(G)_n)$

$\varphi_G(0) = 1$
$\varphi_G(1) = 1$
$\varphi_G(2)$



Age and profile : example on a finite group (1)

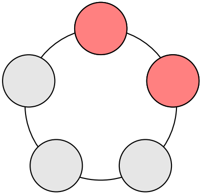
Action of the cyclic group $G = \mathcal{C}_5$ on the five pearl necklace
$\rightarrow$ induced action on *subsets* of pearls

Degree of an orbit : the cardinality shared by all subsets in that orbit

Age of G : $\mathcal{A}(G) = \sqcup_n \mathcal{A}(G)_n, \quad \mathcal{A}(G)_n = \{\text{orbits of degree } n\}$

Profile of G : $\varphi_G : n \mapsto \text{card}(\mathcal{A}(G)_n)$

$\varphi_G(0) = 1$
$\varphi_G(1) = 1$
$\varphi_G(2)$



Age and profile : example on a finite group (1)

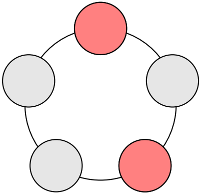
Action of the cyclic group $G = \mathcal{C}_5$ on the five pearl necklace
$\rightarrow$ induced action on *subsets* of pearls

Degree of an orbit : the cardinality shared by all subsets in that orbit

Age of G : $\mathcal{A}(G) = \sqcup_n \mathcal{A}(G)_n, \quad \mathcal{A}(G)_n = \{\text{orbits of degree } n\}$

Profile of G : $\varphi_G : n \mapsto \text{card}(\mathcal{A}(G)_n)$

$\varphi_G(0) = 1$
$\varphi_G(1) = 1$
$\varphi_G(2)$



Age and profile : example on a finite group (1)

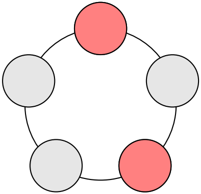
Action of the cyclic group $G = \mathcal{C}_5$ on the five pearl necklace
$\rightarrow$ induced action on *subsets* of pearls

Degree of an orbit : the cardinality shared by all subsets in that orbit

Age of G : $\mathcal{A}(G) = \sqcup_n \mathcal{A}(G)_n, \quad \mathcal{A}(G)_n = \{\text{orbits of degree } n\}$

Profile of G : $\varphi_G : n \mapsto \text{card}(\mathcal{A}(G)_n)$

$\varphi_G(0) = 1$
$\varphi_G(1) = 1$
$\varphi_G(2) = 2$



Age and profile : example on a finite group (1)

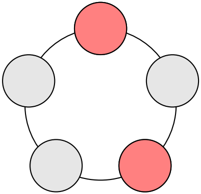
Action of the cyclic group $G = \mathcal{C}_5$ on the five pearl necklace
$\rightarrow$ induced action on *subsets* of pearls

Degree of an orbit : the cardinality shared by all subsets in that orbit

Age of G : $\mathcal{A}(G) = \sqcup_n \mathcal{A}(G)_n$, $\quad \mathcal{A}(G)_n = \{\text{orbits of degree } n\}$

Profile of G : $\varphi_G : n \mapsto \text{card}(\mathcal{A}(G)_n)$

$\varphi_G(0) = 1$
$\varphi_G(1) = 1$
$\varphi_G(2) = 2$
$\varphi_G(3)$



Age and profile : example on a finite group (1)

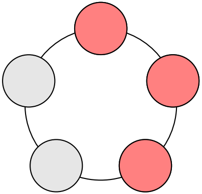
Action of the cyclic group $G = \mathcal{C}_5$ on the five pearl necklace
$\rightarrow$ induced action on *subsets* of pearls

Degree of an orbit : the cardinality shared by all subsets in that orbit

Age of G : $\mathcal{A}(G) = \sqcup_n \mathcal{A}(G)_n, \quad \mathcal{A}(G)_n = \{\text{orbits of degree } n\}$

Profile of G : $\varphi_G : n \mapsto \text{card}(\mathcal{A}(G)_n)$

$\varphi_G(0) = 1$
$\varphi_G(1) = 1$
$\varphi_G(2) = 2$
$\varphi_G(3)$



Age and profile : example on a finite group (1)

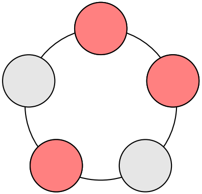
Action of the cyclic group $G = \mathcal{C}_5$ on the five pearl necklace
$\rightarrow$ induced action on *subsets* of pearls

Degree of an orbit : the cardinality shared by all subsets in that orbit

Age of G : $\mathcal{A}(G) = \sqcup_n \mathcal{A}(G)_n, \quad \mathcal{A}(G)_n = \{\text{orbits of degree } n\}$

Profile of G : $\varphi_G : n \mapsto \text{card}(\mathcal{A}(G)_n)$

$\varphi_G(0) = 1$
$\varphi_G(1) = 1$
$\varphi_G(2) = 2$
$\varphi_G(3)$



Age and profile : example on a finite group (1)

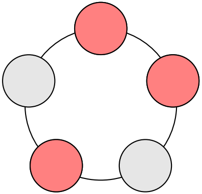
Action of the cyclic group $G = \mathcal{C}_5$ on the five pearl necklace
$\rightarrow$ induced action on *subsets* of pearls

Degree of an orbit : the cardinality shared by all subsets in that orbit

Age of G : $\mathcal{A}(G) = \sqcup_n \mathcal{A}(G)_n, \quad \mathcal{A}(G)_n = \{\text{orbits of degree } n\}$

Profile of G : $\varphi_G : n \mapsto \text{card}(\mathcal{A}(G)_n)$

$\varphi_G(0) = 1$
$\varphi_G(1) = 1$
$\varphi_G(2) = 2$
$\varphi_G(3) = 2$



Age and profile : example on a finite group (1)

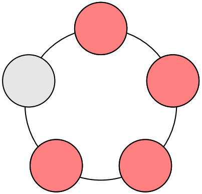
Action of the cyclic group $G = \mathcal{C}_5$ on the five pearl necklace
$\rightarrow$ induced action on *subsets* of pearls

Degree of an orbit : the cardinality shared by all subsets in that orbit

Age of G : $\mathcal{A}(G) = \sqcup_n \mathcal{A}(G)_n, \quad \mathcal{A}(G)_n = \{\text{orbits of degree } n\}$

Profile of G : $\varphi_G : n \mapsto \text{card}(\mathcal{A}(G)_n)$

$\varphi_G(0) = 1$
$\varphi_G(1) = 1$
$\varphi_G(2) = 2$
$\varphi_G(3) = 2$
$\varphi_G(4) = 1$



Age and profile : example on a finite group (1)

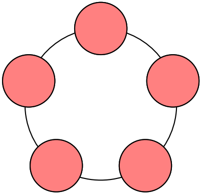
Action of the cyclic group $G = \mathcal{C}_5$ on the five pearl necklace
$\rightarrow$ induced action on *subsets* of pearls

Degree of an orbit : the cardinality shared by all subsets in that orbit

Age of G : $\mathcal{A}(G) = \sqcup_n \mathcal{A}(G)_n, \quad \mathcal{A}(G)_n = \{\text{orbits of degree } n\}$

Profile of G : $\varphi_G : n \mapsto \text{card}(\mathcal{A}(G)_n)$

$\varphi_G(0) = 1$
$\varphi_G(1) = 1$
$\varphi_G(2) = 2$
$\varphi_G(3) = 2$
$\varphi_G(4) = 1$
$\varphi_G(5) = 1$



Age and profile : example on a finite group (1)

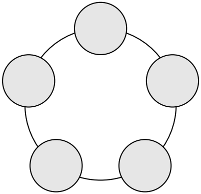
Action of the cyclic group $G = \mathcal{C}_5$ on the five pearl necklace
$\rightarrow$ induced action on *subsets* of pearls

Degree of an orbit : the cardinality shared by all subsets in that orbit

Age of G : $\mathcal{A}(G) = \sqcup_n \mathcal{A}(G)_n, \quad \mathcal{A}(G)_n = \{\text{orbits of degree } n\}$

Profile of G : $\varphi_G : n \mapsto \text{card}(\mathcal{A}(G)_n)$

$\varphi_G(0) = 1$
$\varphi_G(1) = 1$
$\varphi_G(2) = 2$
$\varphi_G(3) = 2$
$\varphi_G(4) = 1$
$\varphi_G(5) = 1$
$\varphi_G(n) = 0 \text{ si } n > 5$



Age and profile : example on a finite group (2)

Generating polynomial of the profile :

$$\mathcal{H}_G(z) = \sum_{n \geq 0} \varphi_G(n) z^n = 1 + z + 2z^2 + 2z^3 + z^4 + z^5$$

Can be calculated using Pólya's theory

Age and profile of infinite permutation groups

- G : a permutation group acting on a countably infinite set E

Age and profile of infinite permutation groups

- G : a permutation group acting on a countably infinite set E
- The generating polynomial becomes a generating series $\mathcal{H}_G$

Age and profile of infinite permutation groups

- G : a permutation group acting on a countably infinite set E
- The generating polynomial becomes a generating series $\mathcal{H}_G$
- The profile may take infinite values

Age and profile of infinite permutation groups

- G : a permutation group acting on a countably infinite set E
- The generating polynomial becomes a generating series $\mathcal{H}_G$
- The profile may take infinite values

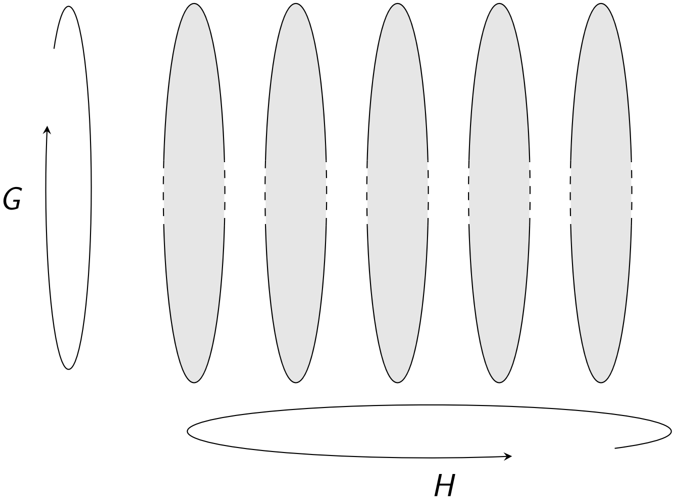
$\rightarrow$ **Oligomorphic permutation groups**:

$$\varphi_G(n) < \infty \quad \forall n \in \mathbb{N}$$

Wreath product of two permutation groups

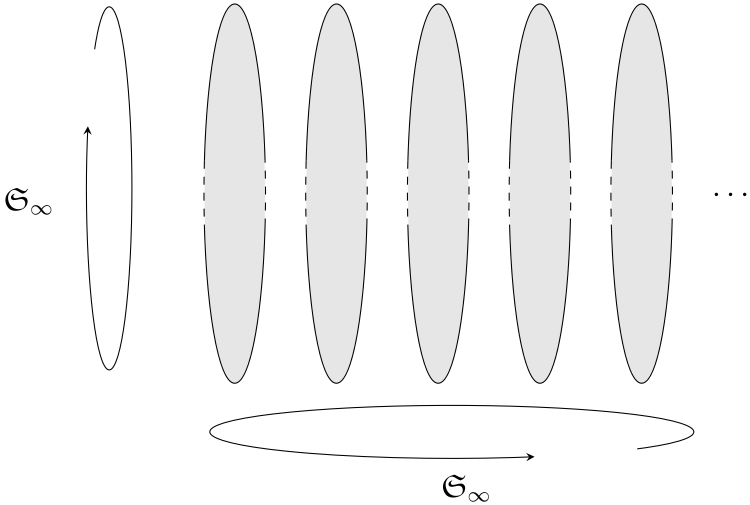
$G \leq \mathfrak{S}_M$, $H \leq \mathfrak{S}_N$

$G \wr H$ has a natural action on $E = \sqcup_{i=1}^{N} E_i$, with $\mathrm{card} E_i = M$.



Example : $G = \mathfrak{S}_\infty \wr \mathfrak{S}_\infty$

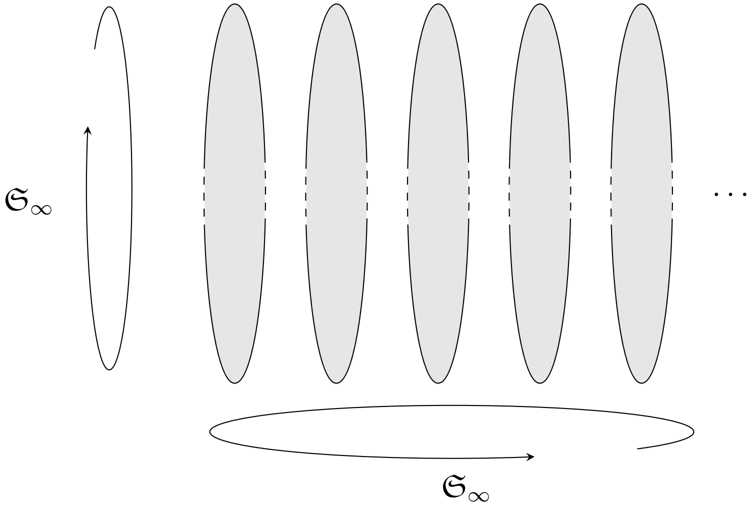
$\varphi_G(n) =$?



Example : $G = \mathfrak{S}_\infty \wr \mathfrak{S}_\infty$

$\varphi_G(n) =$?

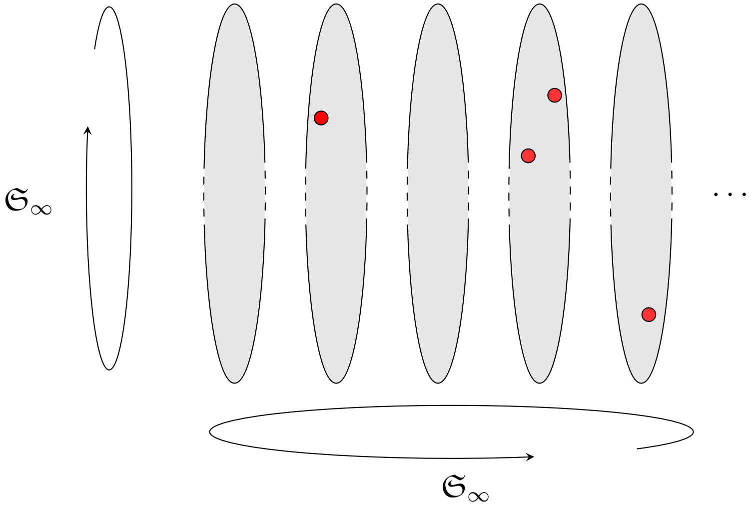
An orbit of degree $n \longleftrightarrow$ a partition of n



Example : $G = \mathfrak{S}_\infty \wr \mathfrak{S}_\infty$

$\varphi_G(n) =$?

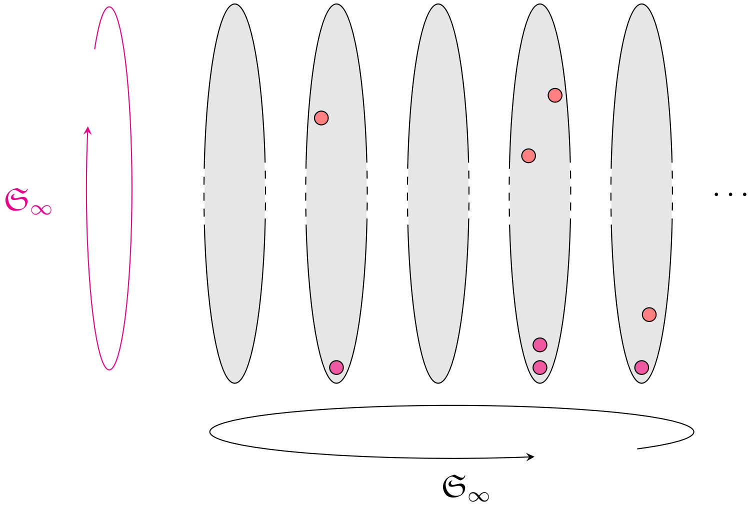
An orbit of degree $n \longleftrightarrow$ a partition of n



Example : $G = \mathfrak{S}_\infty \wr \mathfrak{S}_\infty$

$\varphi_G(n) =$?

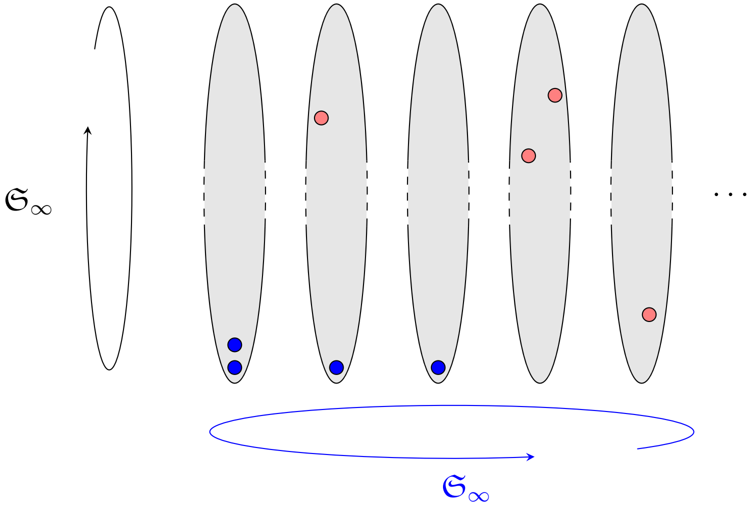
An orbit of degree $n \longleftrightarrow$ a partition of n



Example : $G = \mathfrak{S}_\infty \wr \mathfrak{S}_\infty$

$\varphi_G(n) =$

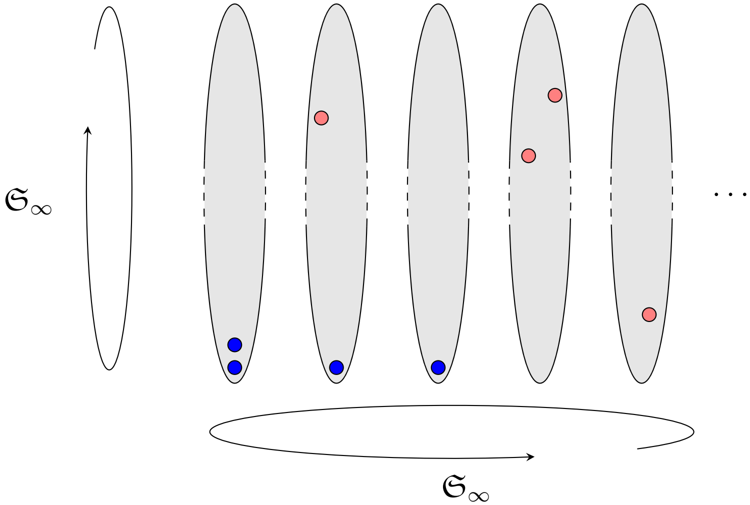
An orbit of degree $n \longleftrightarrow$ a partition of n



Example : $G = \mathfrak{S}_\infty \wr \mathfrak{S}_\infty$

$\varphi_G(n) = \mathscr{P}(n)$

An orbit of degree $n \longleftrightarrow$ a partition of n



Examples

- $G = \mathfrak{S}_\infty \wr \mathfrak{S}_\infty$ (action on a denumerable set of copies of $\mathbb{N}$)
  $\varphi_G(n) = \mathscr{P}(n)$, number of partitions of n

$$\mathcal{H}_G = \frac{1}{\prod_{i=1}^{\infty}(1 - z^i)}$$

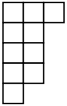
Examples

- $G = \mathfrak{S}_\infty \wr \mathfrak{S}_\infty$ (action on a denumerable set of copies of $\mathbb{N}$)
  $\varphi_G(n) = \mathscr{P}(n)$, number of partitions of n

$$\mathcal{H}_G = \frac{1}{\prod_{i=1}^{\infty}(1 - z^i)}$$

- $G = \mathfrak{S}_m \wr \mathfrak{S}_\infty$
  $\varphi_G(n) = \mathscr{P}_m(n)$, number of partitions into parts of size $\leq m$

$$\mathcal{H}_G = \frac{1}{\prod_{i=1}^{m}(1 - z^i)}$$



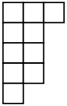
Examples

- $G = \mathfrak{S}_\infty \wr \mathfrak{S}_\infty$ (action on a denumerable set of copies of $\mathbb{N}$)
  $\varphi_G(n) = \mathscr{P}(n)$, number of partitions of n

$$\mathcal{H}_G = \frac{1}{\prod_{i=1}^{\infty}(1 - z^i)}$$

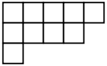
- $G = \mathfrak{S}_m \wr \mathfrak{S}_\infty$
  $\varphi_G(n) = \mathscr{P}_m(n)$, number of partitions into parts of size $\leq m$

$$\mathcal{H}_G = \frac{1}{\prod_{i=1}^{m}(1 - z^i)}$$



- $G = \mathfrak{S}_\infty \wr \mathfrak{S}_m$
  $\varphi_G(n) = \mathscr{P}_m(n)$, number of partitions into at most m parts

$$\mathcal{H}_G = \frac{1}{\prod_{i=1}^{m}(1 - z^i)}$$



Conjecture of Cameron

Conjecture (Cameron, 70s)

If a profile is bounded by a polynomial it is **quasi-polynomial**:

$$\varphi_G(n) = a_s(n)n^s + \cdots + a_1(n)n + a_0(n),$$

where the a_i's are periodic functions.

Conjecture of Cameron

Conjecture (Cameron, 70s)

If a profile is bounded by a polynomial it is **quasi-polynomial**:

$$\varphi_G(n) = a_s(n)n^s + \cdots + a_1(n)n + a_0(n),$$

where the a_i's are periodic functions.

Note

$$\mathcal{H}_G = \frac{P(z)}{(1-z^{d_1})\cdots(1-z^{d_k})} \implies \varphi_G \text{ quasi-polynomial of degree at most } k-1$$

Graded algebras

Definition: Graded algebra

$A = \oplus_n A_n$ such that $A_i A_j \subseteq A_{i+j}$.

Example

$A = \mathbb{K}[x_1, \ldots, x_m]$ is a graded algebra.

A_n: homogeneous polynomials of degree n

Graded algebras

Definition: Graded algebra

$A = \oplus_n A_n$ such that $A_i A_j \subseteq A_{i+j}$.

Example

$A = \mathbb{K}[x_1, \dots, x_m]$ is a graded algebra.
A_n: homogeneous polynomials of degree n

Hilbert series

$\text{Hilbert}(A) = \sum_n \dim(A_n) z^n$

Graded algebras

Definition: Graded algebra

$A = \oplus_n A_n$ such that $A_i A_j \subseteq A_{i+j}$.

Example

$A = \mathbb{K}[x_1, \dots, x_m]$ is a graded algebra.
A_n: homogeneous polynomials of degree n

Hilbert series

$\text{Hilbert}\,(A) = \sum_n \dim(A_n) z^n$

Proposition

A is finitely generated $\implies$ $\text{Hilbert}\,(A) = \frac{P(z)}{(1-z^{d_1})\cdots(1-z^{d_k})}$

Example

$\text{Hilbert}\,(\mathbb{Q}[x, y, t^3]) = \frac{1}{(1-z)^2(1-z^3)}$

A strategy to prove Cameron's conjecture?

- G: an oligomorphic permutation group with polynomial profile
- Find a graded algebra $A(G) = \oplus_{n\geq 0} A_n$ such that

$$\mathcal{H}_G = \mathrm{Hilbert}\left(A(G)\right)$$

A strategy to prove Cameron's conjecture?

- G: an oligomorphic permutation group with polynomial profile
- Find a graded algebra $A(G) = \oplus_{n\geq 0} A_n$ such that

$$\mathcal{H}_G = \text{Hilbert}\,(A(G))$$

- Try to show that $A(G)$ is finitely generated

A strategy to prove Cameron's conjecture?

- G: an oligomorphic permutation group with polynomial profile
- Find a graded algebra $A(G) = \oplus_{n\geq 0} A_n$ such that

$$\mathcal{H}_G = \text{Hilbert}\left(A(G)\right)$$

- Try to show that $A(G)$ is finitely generated
- Deduce:

$$\mathcal{H}_G = \frac{P(z)}{(1-z^{d_1})\cdots(1-z^{d_k})}$$

and thus the quasi-polynomiality of $\varphi_G(n)$

Cameron, 1980: the orbit algebra $\mathbb{Q}\mathcal{A}(G)$

- a commutative connected graded algebra $\mathbb{Q}\mathcal{A}(G) = \oplus_{n\geq 0} A_n$
- $\dim(A_n) = \varphi_G(n)$

Cameron, 1980: the orbit algebra $\mathbb{Q}\mathcal{A}(G)$

- a commutative connected graded algebra $\mathbb{Q}\mathcal{A}(G) = \oplus_{n \geq 0} A_n$
- $\dim(A_n) = \varphi_G(n)$

Vector space structure

- finite formal linear combinations of orbits (ex: $2o_1 + 5o_2 - o_3$)
- graded by degree, with $\dim(A_n) = \varphi_G(n)$ by construction

Cameron, 1980: the orbit algebra $\mathbb{Q}\mathcal{A}(G)$

- a commutative connected graded algebra $\mathbb{Q}\mathcal{A}(G) = \oplus_{n\geq 0} A_n$
- $\dim(A_n) = \varphi_G(n)$

Vector space structure

- finite formal linear combinations of orbits (ex: $2o_1 + 5o_2 - o_3$)
- graded by degree, with $\dim(A_n) = \varphi_G(n)$ by construction

Product?

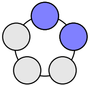
- Defined on subsets:

$$ef = \begin{cases} e \cup f & \text{if } e \cap f = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

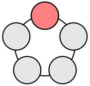
- $o = \{e_1, e_2, \ldots\} \quad \longleftrightarrow \quad e_1 + e_2 + \cdots$

Example of a product in a finite case : back to $\mathcal{C}_5$

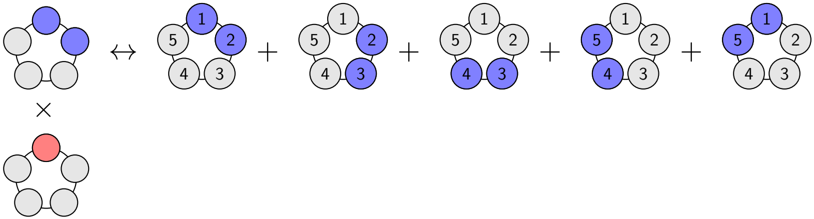
Example of a product in a finite case : back to $\mathcal{C}_5$



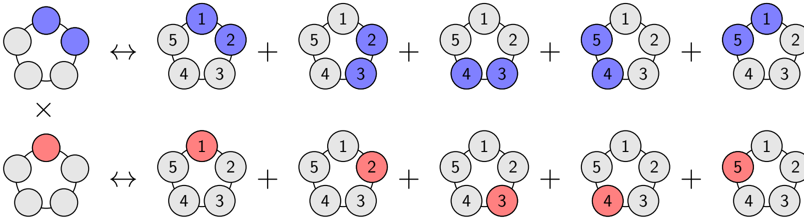
$\times$



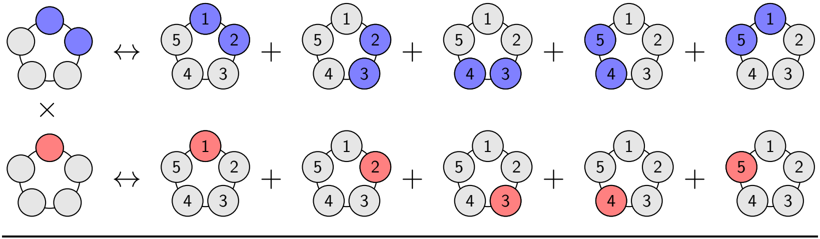
Example of a product in a finite case : back to $\mathcal{C}_5$



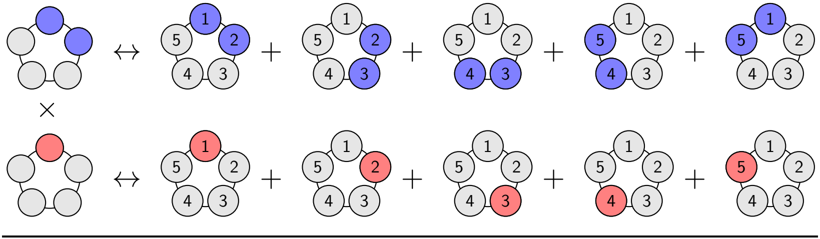
Example of a product in a finite case : back to $\mathcal{C}_5$



Example of a product in a finite case : back to $\mathcal{C}_5$

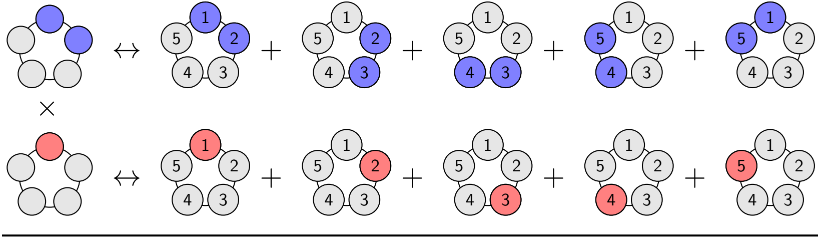


Example of a product in a finite case : back to $\mathcal{C}_5$



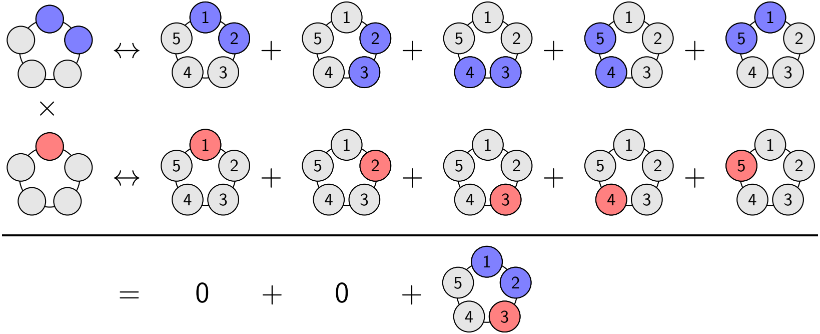
$$= \quad 0$$

Example of a product in a finite case : back to $\mathcal{C}_5$

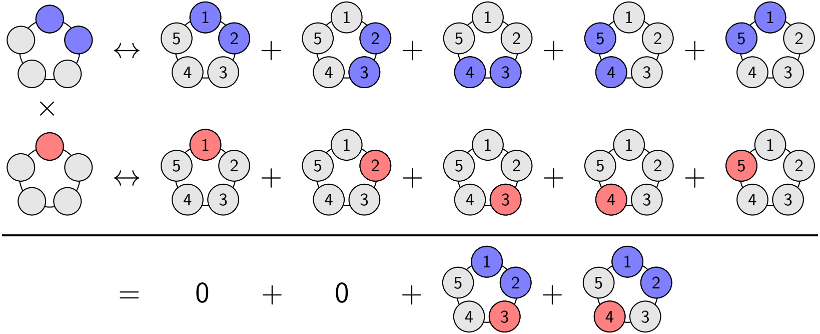


$$= \quad 0 \quad + \quad 0$$

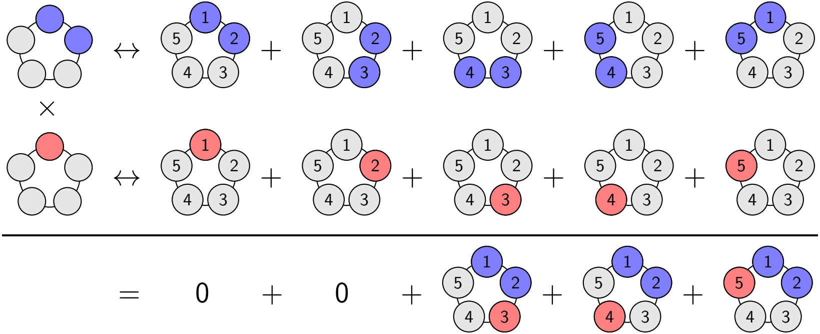
Example of a product in a finite case : back to $\mathcal{C}_5$



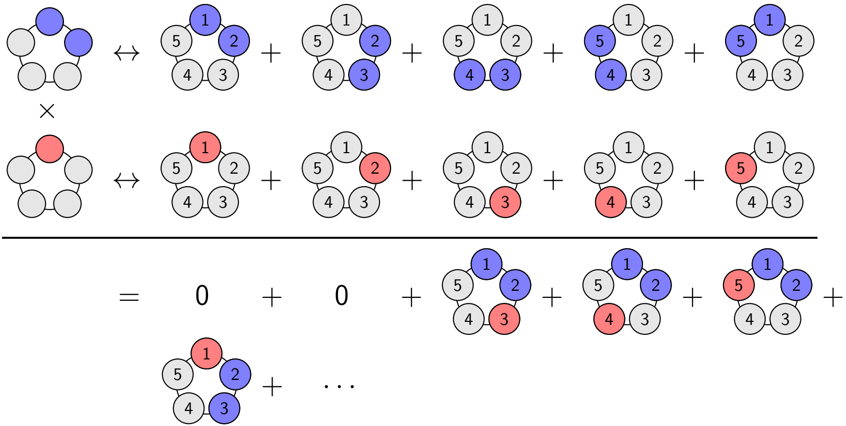
Example of a product in a finite case : back to $\mathcal{C}_5$



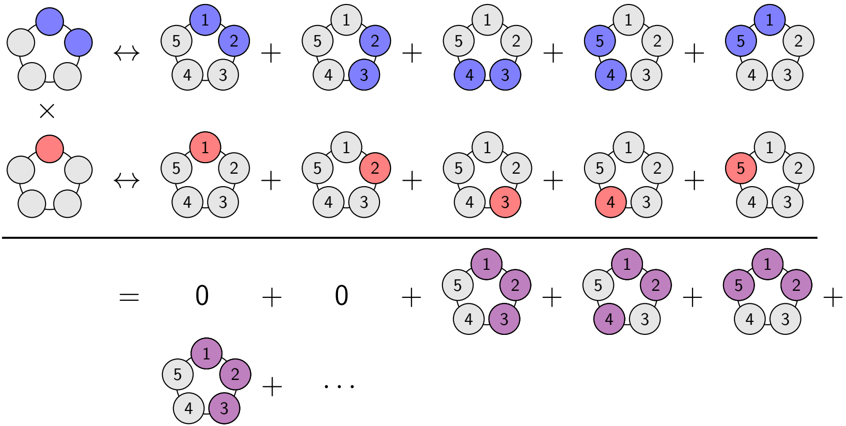
Example of a product in a finite case : back to $\mathcal{C}_5$



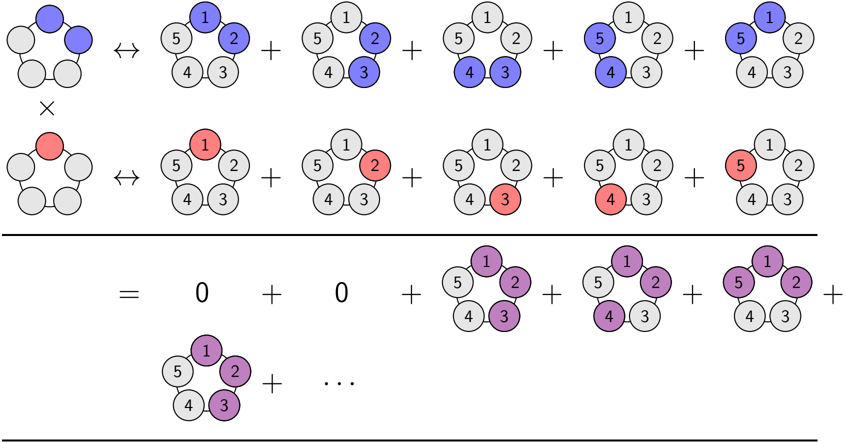
Example of a product in a finite case : back to $\mathcal{C}_5$



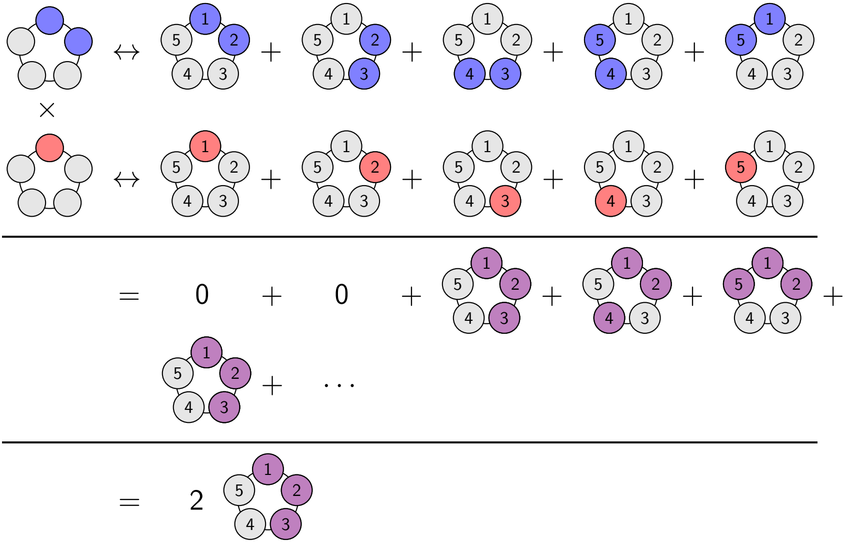
Example of a product in a finite case : back to $\mathcal{C}_5$



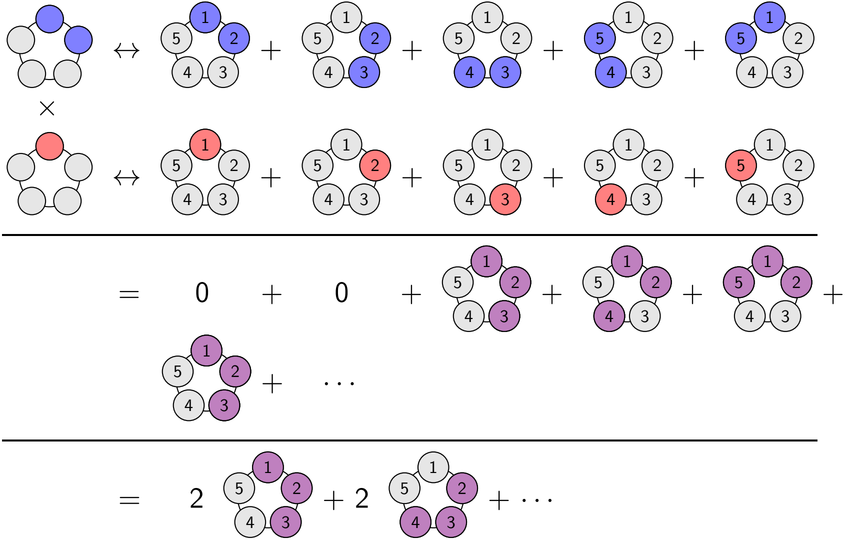
Example of a product in a finite case : back to $\mathcal{C}_5$



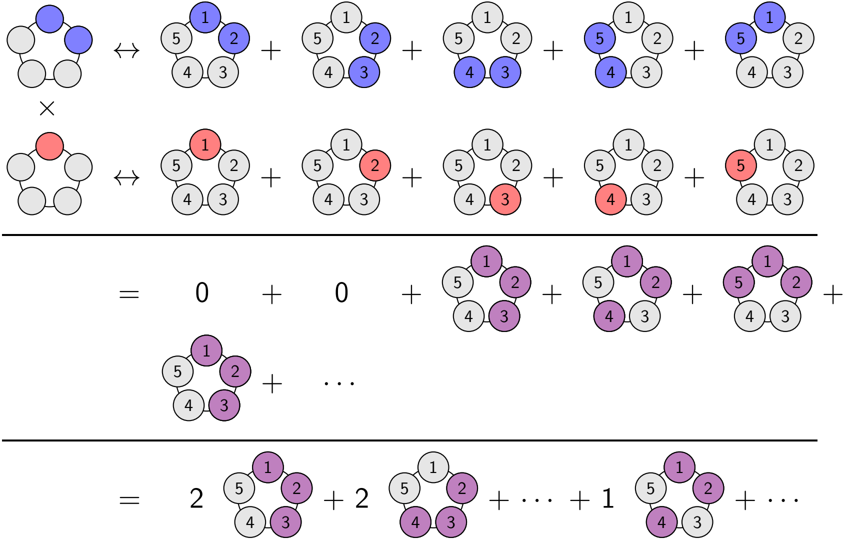
Example of a product in a finite case : back to $\mathcal{C}_5$



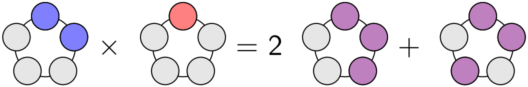
Example of a product in a finite case : back to $\mathcal{C}_5$



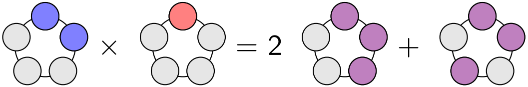
Example of a product in a finite case : back to $\mathcal{C}_5$



In the end:



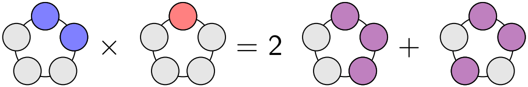
In the end:



Non trivial fact

Product well defined (and graded) on the space of orbits.

In the end:



Non trivial fact

Product well defined (and graded) on the space of orbits.

$\longrightarrow$ **The orbit algebra of a permutation group**

Examples of orbit algebras (1)

Example 1

If $G = \mathfrak{S}_\infty$, $\varphi_G(n) = 1$ for all n, and $\mathbb{Q}\mathcal{A}(G) = \mathbb{K}[x]$.

Examples of orbit algebras (1)

Example 1

If $G = \mathfrak{S}_\infty$, $\varphi_G(n) = 1$ for all n, and $\mathbb{Q}\mathcal{A}(G) = \mathbb{K}[x]$.

Example 2

$G = \mathfrak{S}_\infty \wr \mathfrak{S}_3$, recall that $\varphi_G(n) = \mathscr{P}_3(n)$.

Examples of orbit algebras (1)

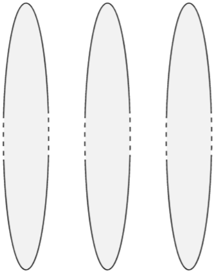
Example 1

If $G = \mathfrak{S}_\infty$, $\varphi_G(n) = 1$ for all n, and $\mathbb{Q}\mathcal{A}(G) = \mathbb{K}[x]$.

Example 2

$G = \mathfrak{S}_\infty \wr \mathfrak{S}_3$, recall that $\varphi_G(n) = \mathscr{P}_3(n)$.

A_n = homogeneous symmetric polynomials of degree n in x_1, x_2, x_3



Examples of orbit algebras (1)

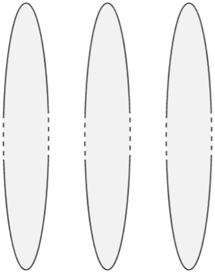
Example 1

If $G = \mathfrak{S}_\infty$, $\varphi_G(n) = 1$ for all n, and $\mathbb{Q}\mathcal{A}(G) = \mathbb{K}[x]$.

Example 2

$G = \mathfrak{S}_\infty \wr \mathfrak{S}_3$, recall that $\varphi_G(n) = \mathscr{P}_3(n)$.

A_n = homogeneous symmetric polynomials of degree n in x_1, x_2, x_3



$\to \mathbb{Q}\mathcal{A}(\mathfrak{S}_\infty \wr \mathfrak{S}_3) = \mathbb{K}[x_1, x_2, x_3]^{\mathfrak{S}_3}$

Examples of orbit algebras (1)

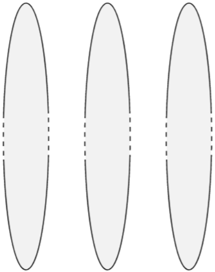
Example 1

If $G = \mathfrak{S}_\infty$, $\varphi_G(n) = 1$ for all n, and $\mathbb{Q}\mathcal{A}(G) = \mathbb{K}[x]$.

Example 2

$G = \mathfrak{S}_\infty \wr \mathfrak{S}_3$, recall that $\varphi_G(n) = \mathscr{P}_3(n)$.

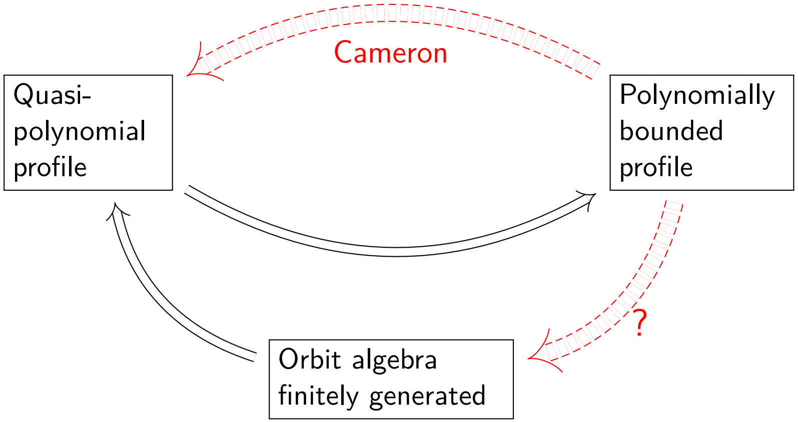
A_n = homogeneous symmetric polynomials of degree n in x_1, x_2, x_3



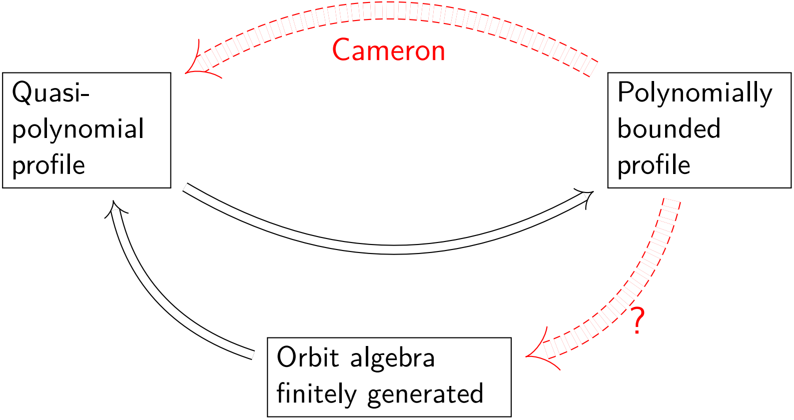
$\to \mathbb{Q}\mathcal{A}(\mathfrak{S}_\infty \wr \mathfrak{S}_3) = \mathbb{K}[x_1, x_2, x_3]^{\mathfrak{S}_3}$

More generally, for H subgroup of $\mathfrak{S}_m$, $\mathbb{Q}\mathcal{A}(\mathfrak{S}_\infty \wr H) = \mathbb{K}[x_1, \ldots, x_m]^H$, the algebra of invariants of H

Overview and conjecture of Macpherson



Overview and conjecture of Macpherson



Conjecture (Macpherson, 1985)

Profile of G polynomial $\Longleftrightarrow$ $\mathbb{Q}\mathcal{A}(G)$ finitely generated

Finite index subgroups

Theorem

Let H be a finite index subgroup of G.

- The profiles of G and H are asymptotically equivalent
- $\mathbb{Q}\mathcal{A}(H)$ finitely generated $\implies$ $\mathbb{Q}\mathcal{A}(G)$ finitely generated

Finite index subgroups

Theorem

Let H be a finite index subgroup of G.

- The profiles of G and H are asymptotically equivalent
- $\mathbb{Q}\mathcal{A}(H)$ finitely generated $\implies$ $\mathbb{Q}\mathcal{A}(G)$ finitely generated

Proof.

Uses invariant theory, and the ideas of the proof of Hilbert's theorem.

Finite index subgroups

Theorem

Let H be a finite index subgroup of G.

- The profiles of G and H are asymptotically equivalent
- $\mathbb{Q}\mathcal{A}(H)$ finitely generated $\implies$ $\mathbb{Q}\mathcal{A}(G)$ finitely generated

Proof.

Uses invariant theory, and the ideas of the proof of Hilbert's theorem.

Application : reduction of Macpherson's conjecture

Without loss of generality, we may assume for instance that G has no finite orbit.

But there will be more...

Block systems

Definition : Block system

Partition of E such that each part is globally mapped onto another one (or itself) by every element of G

(see previous examples)

Block systems

Definition : Block system

Partition of E such that each part is globally mapped onto another one (or itself) by every element of G

(see previous examples)

Relevant notion?

Block systems

Definition : Block system

Partition of E such that each part is globally mapped onto another one (or itself) by every element of G

(see previous examples)

Relevant notion?

Theorem (Macpherson)

If G is **primitive** (i.e. admits no non trivial block system) then G has its profile equal to 1 or exponential.

Block systems

Definition : Block system

Partition of E such that each part is globally mapped onto another one (or itself) by every element of G

(see previous examples)

Relevant notion?

Theorem (Macpherson)

If G is **primitive** (i.e. admits no non trivial block system) then G has its profile equal to 1 or exponential.

$\rightarrow$ The groups we are interested in have a constantly equal to 1 profile or have a block system.

The complete primitive groups

Theorem (Classification, Cameron)

There are exactly 5 complete groups of constantly equal to 1 profile.

The complete primitive groups

Theorem (Classification, Cameron)

There are exactly 5 complete groups of constantly equal to 1 profile.

- $\mathrm{Aut}(\mathbb{Q})$: automorphisms of the rational chain (increasing functions)
- $\mathrm{Rev}(\mathbb{Q})$: generated by $\mathrm{Aut}(\mathbb{Q})$ and one reflection
- $\mathrm{Aut}(\mathbb{Q}/\mathbb{Z})$, preserving the circular order
- $\mathrm{Rev}(\mathbb{Q}/\mathbb{Z})$: generated by $\mathrm{Aut}(\mathbb{Q}/\mathbb{Z})$ and one reflection
- $\mathfrak{S}_\infty$: the symmetric group

The complete primitive groups

Theorem (Classification, Cameron)

There are exactly 5 complete groups of constantly equal to 1 profile.

- $\mathrm{Aut}(\mathbb{Q})$: automorphisms of the rational chain (increasing functions)
- $\mathrm{Rev}(\mathbb{Q})$: generated by $\mathrm{Aut}(\mathbb{Q})$ and one reflection
- $\mathrm{Aut}(\mathbb{Q}/\mathbb{Z})$, preserving the circular order
- $\mathrm{Rev}(\mathbb{Q}/\mathbb{Z})$: generated by $\mathrm{Aut}(\mathbb{Q}/\mathbb{Z})$ and one reflection
- $\mathfrak{S}_\infty$: the symmetric group

Well known, nice groups (called *highly homogeneous*).
In particular, their orbit algebra is finitely generated.

Transitive block systems

Question : which block system should we consider for a given G ?

Transitive block systems

Question : which block system should we consider for a given G ?

Proposition 1

If G is **transitive** on a block system, it has a system of finitely many infinite blocks *or* infinitely many finite blocks of the same size.

Transitive block systems

Question : which block system should we consider for a given G ?

Proposition 1

If G is **transitive** on a block system, it has a system of finitely many infinite blocks *or* infinitely many finite blocks of the same size.

Proposition 2

If N is the finite number (resp. size) of the blocks in this case, then

$$\varphi_G(n) \geq n^{N-1}$$

Transitive block systems

Question : which block system should we consider for a given G ?

Proposition 1

If G is **transitive** on a block system, it has a system of finitely many infinite blocks *or* infinitely many finite blocks of the same size.

Proposition 2

If N is the finite number (resp. size) of the blocks in this case, then

$$\varphi_G(n) \geq n^{N-1}$$

Intuitively : higher lower bound = better description of the group

$\rightarrow$ we want the finite blocks to be big, and the infinite ones to be many (thus small).

Canonical block system

General case : Canonical block system $B(G)$

Finite blocks as big as possible, and some infinite, smallest ones

Canonical block system

General case : Canonical block system $B(G)$

Finite blocks as big as possible, and some infinite, smallest ones

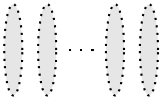
E divided into G-orbits of blocks :

Canonical block system

General case : Canonical block system $B(G)$

Finite blocks as big as possible, and some infinite, smallest ones

E divided into G-orbits of blocks :

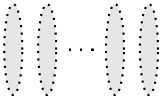


Canonical block system

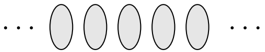
General case : Canonical block system $B(G)$

Finite blocks as big as possible, and some infinite, smallest ones

E divided into G-orbits of blocks :



Or

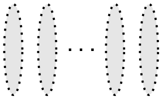


Canonical block system

General case : Canonical block system $B(G)$

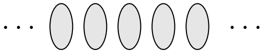
Finite blocks as big as possible, and some infinite, smallest ones

E divided into G-orbits of blocks :



- Reduction $\rightarrow$ *stable* blocks (union of orbits)

Or

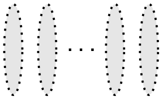


Canonical block system

General case : Canonical block system $B(G)$

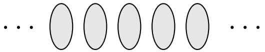
Finite blocks as big as possible, and some infinite, smallest ones

E divided into G-orbits of blocks :



- Reduction $\rightarrow$ *stable* blocks (union of orbits)
- $B(G)$ $\rightarrow$ restrictions are primitive groups

Or

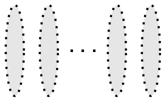


Canonical block system

General case : Canonical block system $B(G)$

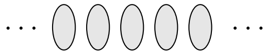
Finite blocks as big as possible, and some infinite, smallest ones

E divided into G-orbits of blocks :



- Reduction $\rightarrow$ *stable* blocks (union of orbits)
- $B(G)$ $\rightarrow$ restrictions are primitive groups

Or



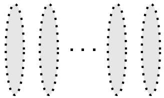
- $B(G)$ $\rightarrow$ action on the blocks is primitive

Canonical block system

General case : Canonical block system $B(G)$

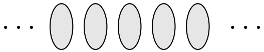
Finite blocks as big as possible, and some infinite, smallest ones

E divided into G-orbits of blocks :



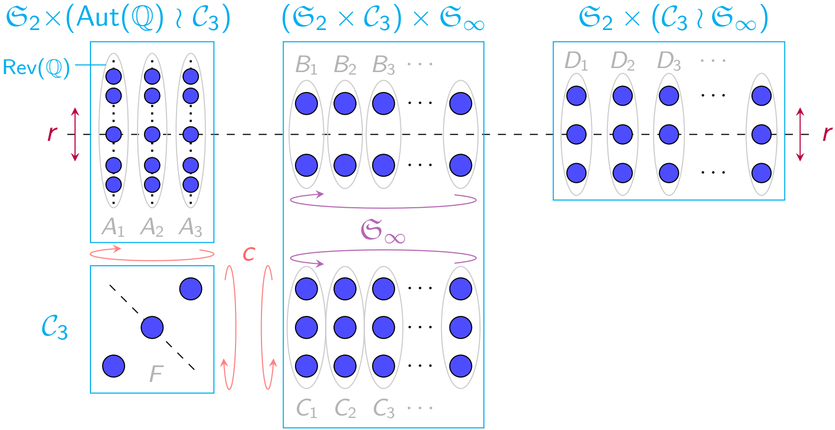
- Reduction $\rightarrow$ *stable* blocks (union of orbits)
- $B(G)$ $\rightarrow$ restrictions are primitive groups

Or



- $B(G)$ $\rightarrow$ action on the blocks is primitive
  Actually, G acts on the blocks as $\mathfrak{S}_\infty$

A typical group with profile bounded by a polynomial



Synchronization

Case of 2 stable parts (ex : 2 infinite orbits)

$E_1 \sqcup E_2$, $G_{|E_1} = G_1, \; G_{|E_2} = G_2$

Synchronization between them ?

Synchronization

Case of 2 stable parts (ex : 2 infinite orbits)

$E_1 \sqcup E_2$, $G_{|E_1} = G_1, \; G_{|E_2} = G_2$

Synchronization between them ?

Evaluated by $G_1/N_1 = G_2/N_2$, where $N_i = \mathrm{Fix}(G, E_j)_{|E_i}$

Synchronization

Case of 2 stable parts (ex : 2 infinite orbits)

$E_1 \sqcup E_2$, $G_{|E_1} = G_1,\ G_{|E_2} = G_2$

Synchronization between them ?

Evaluated by $G_1/N_1 = G_2/N_2$, where $N_i = \mathrm{Fix}(G, E_j)_{|E_i}$

Lemma

The five complete groups of profile 1 have at most one non trivial normal subgroup.

$\rightarrow$ very few possibilities

Synchronization

Case of 2 stable parts (ex : 2 infinite orbits)

$E_1 \sqcup E_2$, $G_{|E_1} = G_1,\ G_{|E_2} = G_2$

Synchronization between them ?

Evaluated by $G_1/N_1 = G_2/N_2$, where $N_i = \text{Fix}(G, E_j)_{|E_i}$

Lemma

The five complete groups of profile 1 have at most one non trivial normal subgroup.

$\rightarrow$ very few possibilities

Example

If $G_1 = G_2 = \mathfrak{S}_\infty$, the actions are either independent or totally synchronized. One may assume safely, for our purposes, the same about the other four groups.

Application to the canonical block system

Works on orbits of blocks $\rightarrow$ no infinite synchronization in $B(G)$

Application to the canonical block system

Works on orbits of blocks → no infinite synchronization in $B(G)$

Convenient fact if $E = E_1 \sqcup E_2$

G_1 and G_2 not synchronized $\Longrightarrow \mathbb{Q}\mathcal{A}(G) = \mathbb{Q}\mathcal{A}(G_1) \otimes \mathbb{Q}\mathcal{A}(G_2)$

Application to the canonical block system

Works on orbits of blocks → no infinite synchronization in $B(G)$

Convenient fact if $E = E_1 \sqcup E_2$

G_1 and G_2 not synchronized $\Longrightarrow \mathbb{Q}\mathcal{A}(G) = \mathbb{Q}\mathcal{A}(G_1) \otimes \mathbb{Q}\mathcal{A}(G_2)$

→ *Orbits of blocks could be treated separately !*

Application to the canonical block system

Works on orbits of blocks → no infinite synchronization in $B(G)$

Convenient fact if $E = E_1 \sqcup E_2$

G_1 and G_2 not synchronized $\Longrightarrow \mathbb{Q}\mathcal{A}(G) = \mathbb{Q}\mathcal{A}(G_1) \otimes \mathbb{Q}\mathcal{A}(G_2)$

→ *Orbits of blocks could be treated separately !*

Last obstacle : remaining finite synchronizations

Orbits of finite blocks

Only essential remaining case : infinite orbits of finite blocks

Orbits of finite blocks

Only essential remaining case : infinite orbits of finite blocks

- Wreath products
  Orbit algebra = free algebra generated by the age of the finite group

Orbits of finite blocks

Only essential remaining case : infinite orbits of finite blocks

- Wreath products $\rightarrow$ OK
  Orbit algebra = free algebra generated by the age of the finite group

Orbits of finite blocks

Only essential remaining case : infinite orbits of finite blocks

- Wreath products $\rightarrow$ OK
  Orbit algebra $=$ free algebra generated by the age of the finite group
- "Direct products" (the finite group acts diagonally)
  Orbit algebra $=$ a certain algebra of invariants

Orbits of finite blocks

Only essential remaining case : infinite orbits of finite blocks

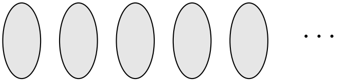
- Wreath products $\rightarrow$ OK
  Orbit algebra = free algebra generated by the age of the finite group
- "Direct products" (the finite group acts diagonally) $\rightarrow$ OK
  Orbit algebra = a certain algebra of invariants

Orbits of finite blocks

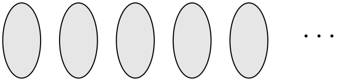
Only essential remaining case : infinite orbits of finite blocks

- Wreath products $\rightarrow$ OK
  Orbit algebra = free algebra generated by the age of the finite group
- "Direct products" (the finite group acts diagonally) $\rightarrow$ OK
  Orbit algebra = a certain algebra of invariants
- General case ?

The "hard case" : transitive block system of finite blocks



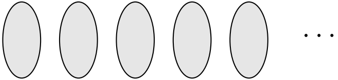
The "hard case" : transitive block system of finite blocks



Definition : Tower of G

H_0 H_1 H_2 ... where H_i is the restriction to the block $i+1$ of the subgroup of G that stabilizes all the blocks and acts trivially on the first i blocks.

The "hard case" : transitive block system of finite blocks



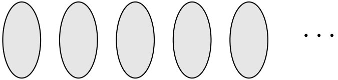
Definition : Tower of G

H_0 H_1 H_2 ... where H_i is the restriction to the block $i+1$ of the subgroup of G that stabilizes all the blocks and acts trivially on the first i blocks.

Proposition 1

Same tower $\implies$ Same orbit algebra

The "hard case" : transitive block system of finite blocks



Definition : Tower of G

$H_0 \; H_1 \; H_2 \; \ldots$ where H_i is the restriction to the block $i+1$ of the subgroup of G that stabilizes all the blocks and acts trivially on the first i blocks.

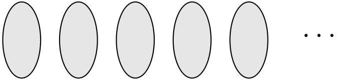
Proposition 1

Same tower $\Longrightarrow$ Same orbit algebra

Proposition 2

The tower of G must be of shape : $H_0 \; H \; H \; H \; \ldots$

The "hard case" : transitive block system of finite blocks



Definition : Tower of G

H_0 H_1 H_2 ... where H_i is the restriction to the block $i+1$ of the subgroup of G that stabilizes all the blocks and acts trivially on the first i blocks.

Proposition 1

Same tower $\Longrightarrow$ Same orbit algebra

Proposition 2

The tower of G must be of shape : H_0 H H H ...
Thus, G has the same orbit algebra as $\frac{H_0}{H} \times H \wr \mathfrak{S}_\infty$,
which is of finite index over $H \wr \mathfrak{S}_\infty$.

The "hard case" : transitive block system of finite blocks

Sketch of proof.

1. Finite case of four blocks only :
   G has tower $H_0\ H_1\ H_2\ H_3 \Rightarrow H_1 = H_2$

The "hard case" : transitive block system of finite blocks

Sketch of proof.

1. Finite case of four blocks only :
   G has tower H_0 H_1 H_2 $H_3 \Rightarrow H_1 = H_2$
2. Prove the infinite case by restricting to every subset of four consecutive blocks

The "hard case" : transitive block system of finite blocks

Sketch of proof.

1. Finite case of four blocks only :
   G has tower $H_0\ H_1\ H_2\ H_3 \Rightarrow H_1 = H_2$
2. Prove the infinite case by restricting to every subset of four consecutive blocks

Conclusion about this case

The "hard case" : transitive block system of finite blocks

Sketch of proof.

1. Finite case of four blocks only :
   G has tower $H_0\ H_1\ H_2\ H_3 \Rightarrow H_1 = H_2$
2. Prove the infinite case by restricting to every subset of four consecutive blocks

Conclusion about this case

- Restrictions to orbits of finite blocks may be thought of as wreath products for the sake of proving the conjecture

The "hard case" : transitive block system of finite blocks

Sketch of proof.

1. Finite case of four blocks only :
   G has tower H_0 H_1 H_2 $H_3 \Rightarrow H_1 = H_2$
2. Prove the infinite case by restricting to every subset of four consecutive blocks

Conclusion about this case

- Restrictions to orbits of finite blocks may be thought of as wreath products for the sake of proving the conjecture
- Solves the issue of possible finite synchronizations between different orbits of blocks

Recap : proof of the conjecture of Macpherson

Recap : proof of the conjecture of Macpherson

1. $B(G)$ canonical block system

Recap : proof of the conjecture of Macpherson

1. $B(G)$ canonical block system
2. Successive reductions to a subgroup of final index

Recap : proof of the conjecture of Macpherson

1. $B(G)$ canonical block system
2. Successive reductions to a subgroup of final index
    - "no" finite orbit of elements

Recap : proof of the conjecture of Macpherson

1. $B(G)$ canonical block system
2. Successive reductions to a subgroup of final index
    - "no" finite orbit of elements
    - infinite blocks are primitive orbits

Recap : proof of the conjecture of Macpherson

1. $B(G)$ canonical block system
2. Successive reductions to a subgroup of final index
   - "no" finite orbit of elements
   - infinite blocks are primitive orbits
   - G acts as a wreath product on the orbits of finite blocks

Recap : proof of the conjecture of Macpherson

1. $B(G)$ canonical block system
2. Successive reductions to a subgroup of final index
   - "no" finite orbit of elements
   - infinite blocks are primitive orbits
   - G acts as a wreath product on the orbits of finite blocks
3. After reduction, only independent orbits of blocks in $B(G)$

Recap : proof of the conjecture of Macpherson

1. $B(G)$ canonical block system
2. Successive reductions to a subgroup of final index
   - "no" finite orbit of elements
   - infinite blocks are primitive orbits
   - G acts as a wreath product on the orbits of finite blocks
3. After reduction, only independent orbits of blocks in $B(G)$
4. $\Rightarrow$ Orbit algebra of the reduced group = a tensor product of algebras of type $\mathbb{K}[x]$, $\mathbb{K}[X]^{G'}$ with some G' finite, and possibly a finite dimensional algebra. Hilbert's theorem and our reduction result conclude.

Recap : proof of the conjecture of Macpherson

1. $B(G)$ canonical block system
2. Successive reductions to a subgroup of final index
   - "no" finite orbit of elements
   - infinite blocks are primitive orbits
   - G acts as a wreath product on the orbits of finite blocks
3. After reduction, only independent orbits of blocks in $B(G)$
4. $\Rightarrow$ Orbit algebra of the reduced group $=$ a tensor product of algebras of type $\mathbb{K}[x]$, $\mathbb{K}[X]^{G'}$ with some G' finite, and possibly a finite dimensional algebra. Hilbert's theorem and our reduction result conclude.

Theorem (Thiéry & F., 2017)

The orbit algebra of an oligomorphic permutation group with profile bounded by a polynomial is finitely generated.

Recap : proof of the conjecture of Macpherson

1. $B(G)$ canonical block system
2. Successive reductions to a subgroup of final index
   - "no" finite orbit of elements
   - infinite blocks are primitive orbits
   - G acts as a wreath product on the orbits of finite blocks
3. After reduction, only independent orbits of blocks in $B(G)$
4. $\Rightarrow$ Orbit algebra of the reduced group $=$ a tensor product of algebras of type $\mathbb{K}[x]$, $\mathbb{K}[X]^{G'}$ with some G' finite, and possibly a finite dimensional algebra. Hilbert's theorem and our reduction result conclude.

Theorem (Thiéry & F., 2017)

The orbit algebra of an oligomorphic permutation group with profile bounded by a polynomial is finitely generated.

In other words, *the conjectures of Macpherson and Cameron hold !*

Stronger result : Cohen-Macaulay algebra

- Finite generation of the orbit algebra $\Rightarrow \mathcal{H}_G = \frac{P(z)}{(1-z^{d_1})\cdots(1-z^{d_k})}$

Stronger result : Cohen-Macaulay algebra

- Finite generation of the orbit algebra $\Rightarrow \mathcal{H}_G = \frac{P(z)}{(1-z^{d_1})\cdots(1-z^{d_k})}$
- Case of Cohen-Macaulay algebras (free finite module over a free finitely generated algebra) : $\exists$ $P(z)$ with positive coefficients

Stronger result : Cohen-Macaulay algebra

- Finite generation of the orbit algebra $\Rightarrow \mathcal{H}_G = \frac{P(z)}{(1-z^{d_1})\cdots(1-z^{d_k})}$
- Case of Cohen-Macaulay algebras (free finite module over a free finitely generated algebra) : $\exists\ P(z)$ with positive coefficients
- Once again, it is possible to adapt a proof of invariant theory to obtain that the orbit algebra is indeed a Cohen-Macaulay algebra

Thank you for your attention !

Context

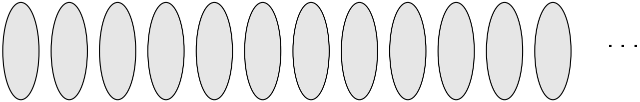
- G permutation group of a countably infinite set E
- Profile φ_G: counts the orbits of finite subsets of E
- **Hypothesis** : $\varphi_G(n)$ bounded by a polynomial
- Conjecture (Cameron) : quasi-polynomiality of φ_G
- Conjecture (Macpherson) : finite generation of the orbit algebra

Results

- Both conjectures hold !
- The orbit algebra is a Cohen-Macaulay algebra
- Classification of the P-oligomorphic permutation groups...

Direct product in the case of finite blocks

"Speak, friend..."

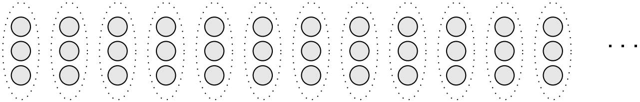


Direct product in the case of finite blocks

"Speak, friend..."

Example 3

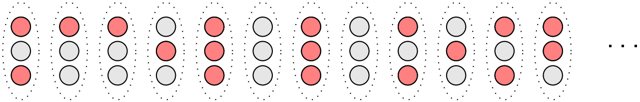
$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3



Direct product in the case of finite blocks

Example 3

$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3

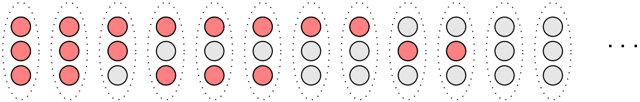


Direct product in the case of finite blocks

"Speak, friend..."

Example 3

$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3

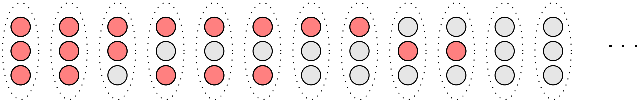


Direct product in the case of finite blocks

"Speak, friend..."

Example 3

$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3

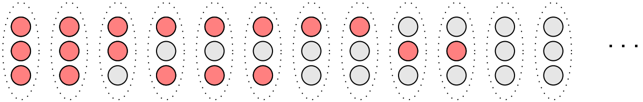


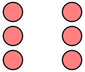
Direct product in the case of finite blocks

"Speak, friend..."

Example 3

$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3



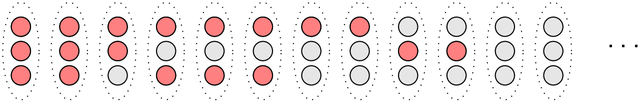


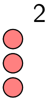
Direct product in the case of finite blocks

"Speak, friend..."

Example 3

$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3



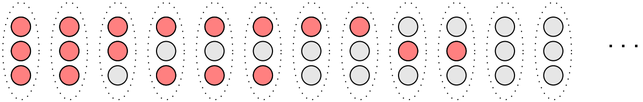


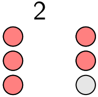
Direct product in the case of finite blocks

"Speak, friend..."

Example 3

$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3



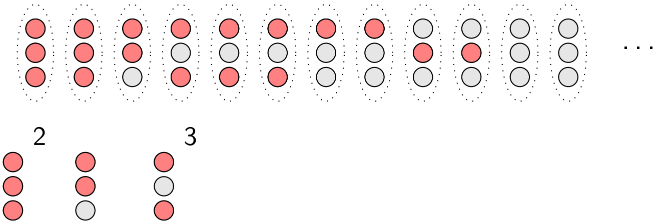


Direct product in the case of finite blocks

"Speak, friend..."

Example 3

$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3

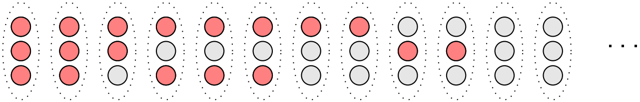


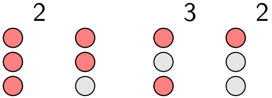
Direct product in the case of finite blocks

"Speak, friend..."

Example 3

$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3



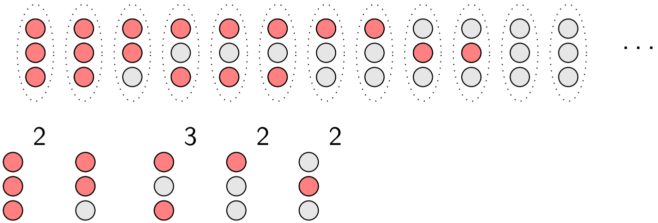


Direct product in the case of finite blocks

"Speak, friend..."

Example 3

$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3

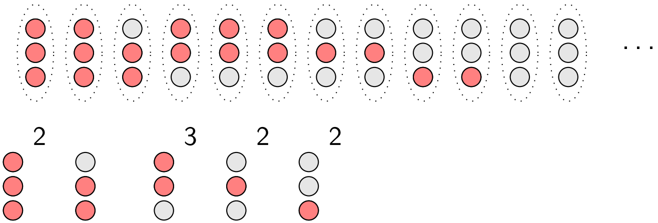


Direct product in the case of finite blocks

"Speak, friend..."

Example 3

$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3

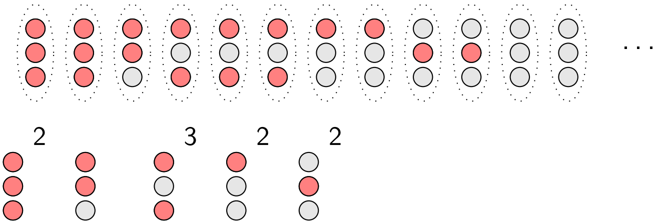


Direct product in the case of finite blocks

"Speak, friend..."

Example 3

$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3

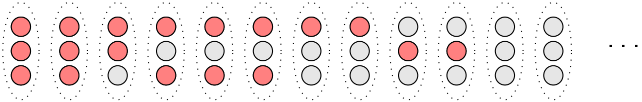


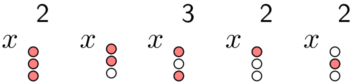
Direct product in the case of finite blocks

"Speak, friend..."

Example 3

$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3



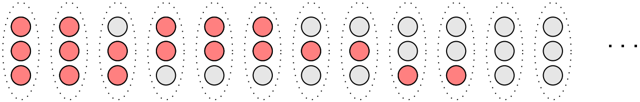


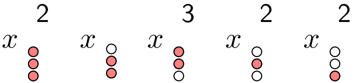
Direct product in the case of finite blocks

"Speak, friend..."

Example 3

$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3



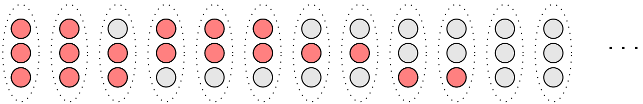


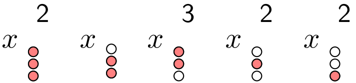
Direct product in the case of finite blocks

"Speak, friend..."

Example 3

$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3





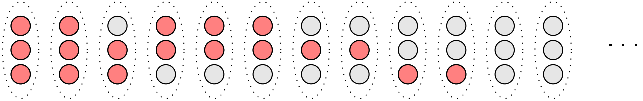
$\rightarrow$ C_3 acts on monomials

Direct product in the case of finite blocks

"Speak, friend..."

Example 3

$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3



$G' = C_3$ acting on (non empty) subsets

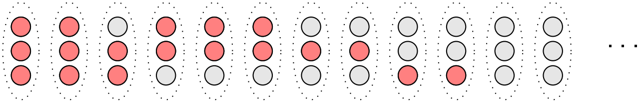
$\mathbb{K}[\ x\ ]^{G'} \quad \longleftrightarrow \quad$ Orbit algebra of $C_3 \times \mathfrak{S}_\infty$?

Direct product in the case of finite blocks

"Speak, friend..."

Example 3

$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3



$G' = C_3$ *acting on (non empty) subsets*

$\mathbb{K}[\ x\ ]^{G'} \quad \longleftrightarrow \quad$ Orbit algebra of $C_3 \times \mathfrak{S}_\infty$?

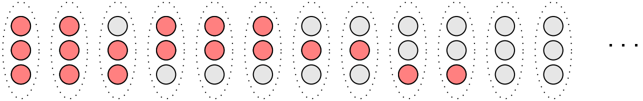
x 

x 

Direct product in the case of finite blocks "Speak, friend..."

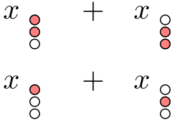
Example 3

$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3



$G' = C_3$ acting on (non empty) subsets

$\mathbb{K}[\ x\ ]^{G'} \quad \longleftrightarrow \quad$ Orbit algebra of $C_3 \times \mathfrak{S}_\infty$?

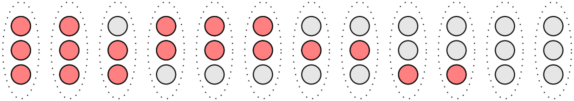


Direct product in the case of finite blocks

"Speak, friend..."

Example 3

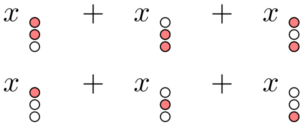
$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3



. . .

$G' = C_3$ acting on (non empty) subsets

$\mathbb{K}[\ x\ ]^{G'} \quad \longleftrightarrow \quad$ Orbit algebra of $C_3 \times \mathfrak{S}_\infty$?

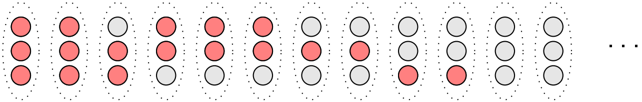


Direct product in the case of finite blocks

"Speak, friend..."

Example 3

$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3



$G' = C_3$ acting on (non empty) subsets

$\mathbb{K}[\, x \,]^{G'} \quad \longleftrightarrow \quad$ Orbit algebra of $C_3 \times \mathfrak{S}_\infty$?

O(x )

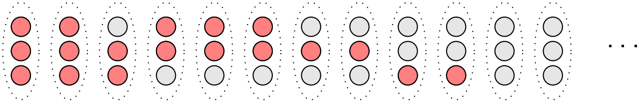
O(x )

Direct product in the case of finite blocks

"Speak, friend..."

Example 3

$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3



$G' = C_3$ acting on (non empty) subsets

$\mathbb{K}[\ x\ ]^{G'}$ $\longleftrightarrow$ Orbit algebra of $C_3 \times \mathfrak{S}_\infty$?

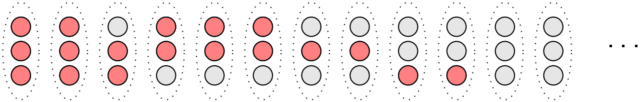
O(x ).O(x )

Direct product in the case of finite blocks

"Speak, friend..."

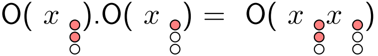
Example 3

$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3



$G' = C_3$ *acting on (non empty) subsets*

$\mathbb{K}[\ x\ ]^{G'} \quad \longleftrightarrow \quad$ Orbit algebra of $C_3 \times \mathfrak{S}_\infty$?

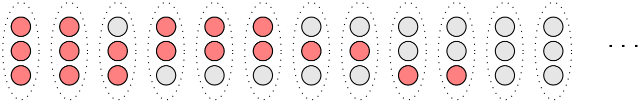


Direct product in the case of finite blocks

"Speak, friend..."

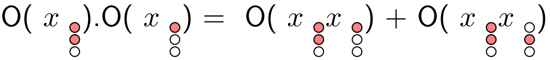
Example 3

$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3



$G' = C_3$ *acting on (non empty) subsets*

$\mathbb{K}[\ x\ ]^{G'} \quad \longleftrightarrow \quad$ Orbit algebra of $C_3 \times \mathfrak{S}_\infty$?

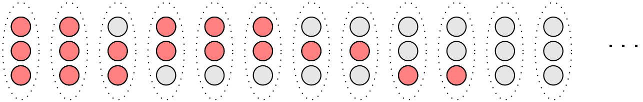


Direct product in the case of finite blocks

"Speak, friend..."

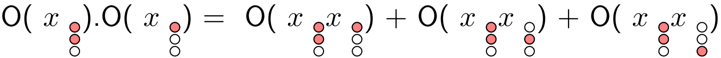
Example 3

$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3



$G' = C_3$ *acting on (non empty) subsets*

$\mathbb{K}[\ x\ ]^{G'} \quad \longleftrightarrow \quad$ Orbit algebra of $C_3 \times \mathfrak{S}_\infty$?

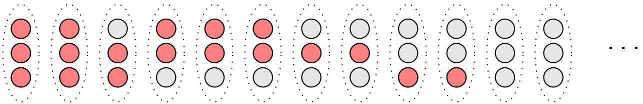


Direct product in the case of finite blocks

"Speak, friend..."

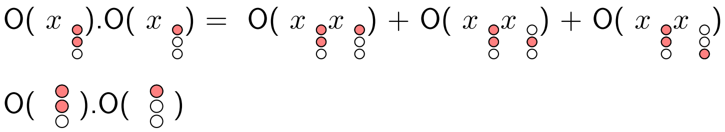
Example 3

$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3



$G' = C_3$ acting on (non empty) subsets

$\mathbb{K}[\,x\,]^{G'} \quad \longleftrightarrow \quad$ Orbit algebra of $C_3 \times \mathfrak{S}_\infty$?

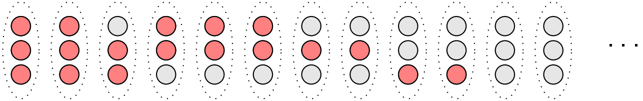


Direct product in the case of finite blocks

"Speak, friend..."

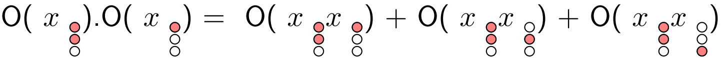
Example 3

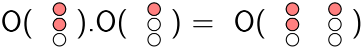
$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3



$G' = C_3$ *acting on (non empty) subsets*

$\mathbb{K}[\,x\,]^{G'} \quad \longleftrightarrow \quad$ Orbit algebra of $C_3 \times \mathfrak{S}_\infty$?



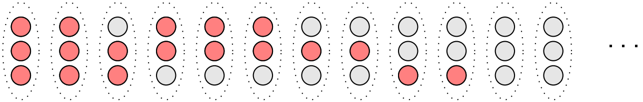


Direct product in the case of finite blocks

"Speak, friend..."

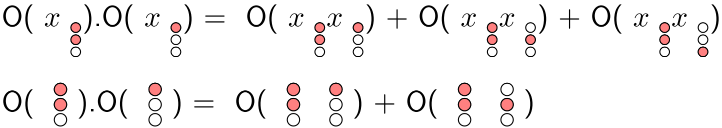
Example 3

$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3



$G' = C_3$ acting on (non empty) subsets

$\mathbb{K}[\,x\,]^{G'} \quad \longleftrightarrow \quad$ Orbit algebra of $C_3 \times \mathfrak{S}_\infty$?

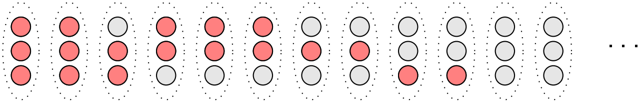


Direct product in the case of finite blocks

"Speak, friend..."

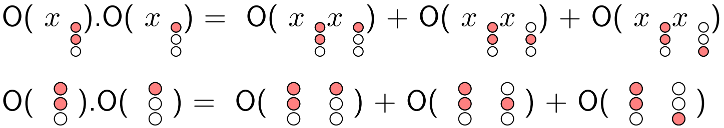
Example 3

$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3



$G' = C_3$ *acting on (non empty) subsets*

$\mathbb{K}[\, x \,]^{G'} \quad \longleftrightarrow \quad$ Orbit algebra of $C_3 \times \mathfrak{S}_\infty$?

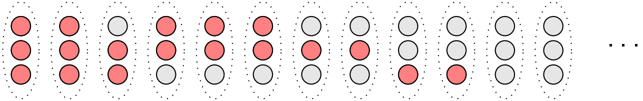


Direct product in the case of finite blocks

"Speak, friend..."

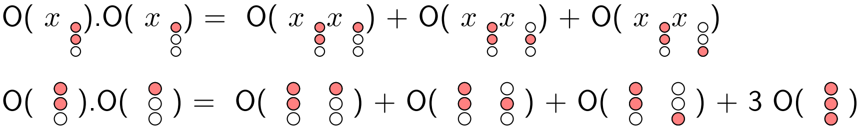
Example 3

$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3



$G' = C_3$ *acting on (non empty) subsets*

$\mathbb{K}[\ x\ ]^{G'} \quad \longleftrightarrow \quad$ Orbit algebra of $C_3 \times \mathfrak{S}_\infty$?

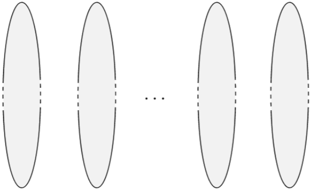


Examples of orbit algebras (2)

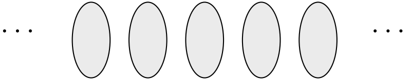
More generally, for H subgroup of $\mathfrak{S}_m$:

- $G = \mathfrak{S}_\infty \wr H$:
  $\mathbb{Q}\mathcal{A}(G) = \mathbb{K}[x_1, \ldots, x_m]^H$, the algebra of invariants of H

  $\mathbb{Q}\mathcal{A}(G)$ is finitely generated by Hilbert's theorem.



- $G = H \wr \mathfrak{S}_\infty$:
  $\mathbb{Q}\mathcal{A}(G) =$ the free algebra generated by the age of H



The "hard" case : case of four blocks

Lemma to prove

G has tower $H_0\ H_1\ H_2\ H_3 \Rightarrow H_1 = H_2$

Lemma

In the general case :
$Fix_G(B_1, \ldots, B_n)$ acts on the remaining blocks as $\mathfrak{S}_\infty$
(due to the absence of normal subgroup of finite index of $\mathfrak{S}_\infty$).

Proof.

An element $s \in G$ stabilizing the blocks $\leftrightarrow$ a quadruple
$g \in H_1 \quad \rightarrow \quad \exists\ (1, g, h, k), \quad h, k \in H_1$.
Let σ be an element of G that permutes the first two blocks and fixes the other two.
Conjugation of x by σ in $G \quad \rightarrow \quad y = (g', 1, h, k)$
Then: $x^{-1}y = (g', g^{-1}, 1, 1)$
By arguing that the tower does not depend on the ordering of the blocks, g^{-1} and therefore g are in H_2.