Pseudo-finite monoids and semigroups

Rida-E Zenab

University of York

NBSAN, 07-08 January 2018

Based on joint work with Victoria Gould and Dandan Yang

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Left congruences

Let M be a monoid and let $\overline{X} \subseteq M \times M$. We denote by $\rho_{\overline{X}}$ the smallest left congruence relation on M containing $\overline{X}$.

For $a, b \in M$, $a\,\rho_{\overline{X}}\,b$ if and only if $a = b$ or there is an $n \geq 1$ and a sequence

$$a = t_1 c_1, t_1 d_1 = t_2 c_2, t_2 d_2 = t_3 c_3, \cdots, t_n d_n = b,$$

where $(c_i, d_i) \in \overline{X} \cup \overline{X}^{-1}$ and $t_i \in M$.

Such a sequence is referred to as an $\overline{X}$-*sequence of length* n. If $n = 0$, we interpret this sequence as being $a = b$.

Definitions

Pseudo-generated monoids

Let M be a monoid and let $X \subseteq M$. Suppose

$$\overline{X} = \{(1, x) : x \in X\} \subseteq M \times M$$

and let $\rho_{\overline{X}}$ be the left congruence on M generated by $\overline{X}$. We say M is pseudo-generated by X if $\rho_{\overline{X}} = \omega_M$.

If X is finite, then M is said to be pseudo-generated by a finite set.

Pseudo-finite monoids

We say M is pseudo-finite, if there is a bound on the length of $\overline{X}$-sequence.

Background: different sources of motivation

Semigroup Algebras

A semigroup algebra $\ell^1(S)$ is the Banach algebra generated by semigroup S.

A weight on a seimgroup S is a function $w : S \to [1, \infty)$ such that

$$w(uv) \leq w(u)w(v) \;\; u, v \in S.$$

Define

$$\ell^1(S, w) = \Big\{ f : S \to \mathbb{C} : \|f\|_w := \sum_{u \in S} |f(u)| w(u) < \infty \Big\}.$$

Then $\ell^1(S, w)$ is a Banach space under pointwise operations with the norm given by $\|\cdot\|_w$ and a Banach algebra if multiplaction is given by convolution. Such a Banach algebra is called weighted semigroup algebra.

Background: different sources of motivation

The augmentation ideal of $\ell^1(S, w)$ is defined as

$$\ell_0^1(S, w) = \left\{ f \in \ell^1(S, w) : \sum_{u \in S} f(u) = 0 \right\}.$$

Theorm (J. T. White)

Let S be a monoid. Then $\ell_0^1(S)$ is finitely-generated if and only if S is pseudo-finite

White's Motivation was

Dales-Zelazko conjecture

Let A be a unital Banach algebra in which every maximal left ideal is finitely-generated. Then A is finite dimensional.

The above conjecture has answer for the class of Banach algebras $\ell^1(M)$ where M is *weakly right cancellative monoid*, but remains open for an arbitrary monoid.

Background: different sources of motivation

Ancestry for a monoid

Let M be a monoid with identity 1 and let $X \subseteq M$. A finite sequence $(z_i)_{i=1}^{n}$ of elements in M is called an ancestry for $m \in M$ of length n with respect to X if $z_1 = m$, $z_n = 1$ and for each $i \in \mathbb{N}$ with $1 < i \leq n$ there exists $x \in X$ such that either $z_i x = z_{i-1}$ or $z_i = z_{i-1} x$.

Lemma

A monoid M is pseudo-finite if and only if there is a finite set X such that every element of M has an ancestry of bounded length with respect to X.

Lemma

A monoid M is pseudo-generated by a finite set X if and only if every element of M has an ancestry with respect to X.

Background: different sources of motivation

Kobayashi's criterion, the property left-FP_1 and Cayley graphs

The property Left-FP_n for monoids

Let M be a monoid and $\mathbb{Z}M$ be the monoid ring over the integers $\mathbb{Z}$. For $n \geq 0$, M is of tpye left-FP_n if there is a resolution

$$A_n \to A_{n-1} \to \cdots \to A_1 \to A_0 \to \mathbb{Z} \to 0$$

of the trivial left $\mathbb{Z}M$-module $\mathbb{Z}$ such that $A_0, A_1, \cdots, A_n$ are finitely-generated left $\mathbb{Z}M$-modules.

Monoids of type right-FP_n are defined dually.
For $n = 1$, a group is of type FP_1 if and only if it is finitely-generated.
This is not case for monoids.
The property left-FP_1 for monoids is characterised by Kobayashi.

Background: Kobayashi's criterion

Right unitary monoids

A submonoid N of a monoid M is said to be right unitary if $mn \in N$ implies $m \in N$ for any $n \in N$ and $m \in M$.
For a subset X of M, $U^r(X)$ denotes the smallest right unitary submonoid of M containing X.
If $M = U^r(X)$, then M is said to be right unitarily generated by X.
If X is finite, then M is said to be right unitarily finitely generated by X.

Cayley graphs

Let M be a monoid and X be a subset of M. The right Cayley graph $\Gamma(M, X)$ of M with respect to X is the directed labelled graph with vertices the elements of M, and a directed edge from p to q labelled by $x \in X$ if and only if $px = q$ in M.
If there is an undirected path between any two vertices, then we say that $\Gamma(M, X)$ is connected.

Background: Kobayashi's criterion

Theorem (Y. Kobayashi 2006)

A monoid M is of type left-FP_1 if and only if there is a finite subset X of M such that one of the following equivalent conditions is satisfied:

1. M is right unitarily generated by X;
2. the right Cayley graph $\Gamma(M, X)$ is connected.

Pseudo-finite/Pseudo-generated monoids and Kobayashi's criterion

if M is a monoid pseudo-generated by a finite set X, then the smallest left congruence $\rho_{\overline{X}} = \langle\{(1, x) : x \in X\}\rangle$ is completely determined by the set $A = \{m \in M : (1, m) \in \rho_{\overline{X}}\}$. Clearly A is a submonoid of M and is right unitary because for any $a \in M$ and $b \in A$

$$ab \in A \Rightarrow a \in A.$$

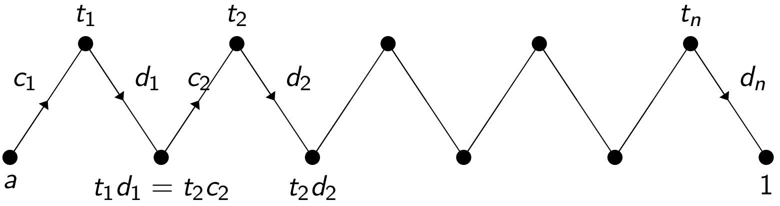
Thus M is right unitarily generated by X

Pseudo-finite/Pseudo-generated monoids and Kobayashi's criterion

Also For any $m \in M$, there is a sequence

$$m = t_1 c_1, t_1 d_1 = t_2 c_2, \cdots, t_n d_n = 1$$

where $(c_i, d_i) \in \overline{X} \cup \overline{X}^{-1}$ and $t_i \in M$ for $1 \leq i \leq n$.
This gives us a path



so that $\Gamma(M, X)$ is connected.

Pseudo-finite/Pseudo-generated monoids and Kobayashi's criterion

Theorem

Let M be a monoid and X be a finite subset of M. Then the following are equivalent:

1. M is pseudo-generated by X;
2. each element of M has an ancestry with respect to X;
3. M is right unitarily finitely generated by X;
4. M is of type left FP_1;
5. the right Cayley graph $\Gamma(M, X)$ of M with respect to X is connected.

Which monoids are pseudo-generated by a finite set/pseudo-finite?

Groups

- Let G be a group and X be a (finite) subset of G. Then G is (finitely) generated by X if and only if G is pseudo-generated by X.
- A group G is pseudo-finite if and only if G is finite.

Finite monoids

Finite monoids are pseudo-finite.

Monoids with zero

Any monoid with zero is pseudo-finite.

Which monoids are pseudo-generated by a finite set/pseudo-finite?

Monoid semilattices

Let $\mathcal{Y}$ be a semilattice with identity 1. Then the following are equivalent:

1. $\mathcal{Y}$ is pseudo-generated by some finite set;
2. $\mathcal{Y}$ has a zero;
3. $\mathcal{Y}$ is pseudo-finite.

Which monoids are pseudo-generated by a finite set/pseudo-finite?

Homomorphic images, retracts and direct products

1. The homomorphic image (retract) of a monoid pseudo-generated by a finite set X is pseudo-generated by a finite set.
2. The homomorphic image (retract) of a pseudo-finite monoid is pseudo-finite.
3. Let S and T be monoids. Then S and T are pseudo-generated by some finite sets X and Y respectively if and only if $S \times T$ is pseudo-generated by $X \times Y$.
4. The direct product of monoids S and T is pseudo-finite if and only if S and T are pseudo-finite.

Which monoids are pseudo-generated by a finite set/pseudo-finite?

Inverse monoids

Suppose S is an inverse monoid with semilattice of idempotents $E(S)$. Then S is pseudo-finite if and only if $E(S)$ has a zero and the corresponding group $\mathcal{H}$-class is finite.

Bruck-Reilly extension of a monoid

Let S be a monoid with identity e. Suppose S is pseudo-generated by a finite set X. Then the Bruck-Reilly extension $T = BR(S, \theta)$ of S determined by θ is pseudo-generated by a finite set

$$X' = \{(1, e, 0), (0, e, 0), (0, x_i, 0) : x_i \in X\}.$$

Which monoids are pseudo-generated by a finite set/pseudo-finite?

Bicyclic Monoid

The Bicyclic monoid $\mathbb{N}^0 \times \mathbb{N}^0$ is pseudo-generated by a finite set

$$X = \{(1,0),(0,0)\}.$$

Rectangular bands

Let B^1 be a rectangular band with an identity adjoined and let X be a finite subset of B^1. Then B^1 is pseudo-generated by X if and only if B^1 has finitely many $\mathcal{R}$-classes.

Which monoids are pseudo-generated by a finite set/pseudo-finite?

Strong semilattices of semigroups

1. Let $S = [\mathcal{Y}; S_\alpha; \phi_{\alpha,\beta}]$ be a strong semilattice of semigroups. Then S^1 is pseudo-generated by a finite set X if and only if $\mathcal{Y}^1$ has a zero and S_0^1 is pseudo-generated.
2. Suppose $\mathcal{N} = [\mathcal{Y}; S_\alpha; \phi_{\alpha,\beta}]$ is a normal band. Then $\mathcal{N}^1$ is pseudo-generated by a finite set X if and only if $\mathcal{Y}^1$ has a zero and S_0^1 has finitely many $\mathcal{R}$-classes.
3. Let $S = [\mathcal{Y}; G_\alpha; \phi_{\alpha,\beta}]$ be a Clifford monoid. Then S is pseudo-generated by some finite set (S is pseudo-finite) if and only if $\mathcal{Y}$ has a 0 and G_0 is finitely generated (finite).

Which monoids are pseudo-generated by a finite set/pseudo-finite?

Finite Rees Index

Let S be a semigroup and T be a subsemigroup of S. The *Rees index* of T in S is defined to be the cardinality of the compliment $S \setminus T$.

Theorem

Let S be a monoid and suppose T be a retract of S having finite Rees index. Then S is pseudo-generated by a finite set if and only if T is pseudo-generated by some finite set.

Counter example to Dales/White conjecture

Dales/White conjecture (in an informal discussion)

A monoid is pseudo-finite if and only if it is direct product of a monoid with zero by a finite monoid.

Example

Let $\mathcal{Y} = \{\alpha, \beta\}$ be a semilattice with $\beta < \alpha$ and let $M = [\mathcal{Y}; G_\alpha; \phi_{\alpha,\beta}]$ be a strong semilattice of groups, where $G_\alpha = G$ is an infinite group with identity 1 and no elements of G has order 2, $G_\beta = \{a, e\}$ is a group with identity e, and $\phi_{\alpha,\beta} : G_\alpha \to G_\beta$ is defined by $g\phi_{\alpha,\beta} = e$ for all $g \in G$. Then M is an infinite pseudo-finite monoid without zero, and it is impossible to be isomorphic to a direct product of a monoid with zero by a finite monoid.

What can we say for semigroups?

Pseudo-generated semigroups

Let S be a semigroup and let $X \subseteq S$. Let $\overline{X} = \{(x, y) : x, y \in X\}$. We say that S is pseudo-generated by X if $\omega_S = \rho_{\overline{X}}$, where $\rho_{\overline{X}}$ is the smallest left congruence relation on S containing $\overline{X}$.

Lemma

Let S be a semigroup and ω_S be finitely generated by $H \subseteq S \times S$. Suppose $\omega_S = \langle K \rangle$ for some $K \subseteq S \times S$. Then there exists a finite subset K' of K such that $\omega_S = \langle K' \rangle$.

Further, if there exists $m \in \mathbb{N}$ such that for any $a, b \in S$, there is a H-sequence from a to b of length at most m, then there is an $m' \in \mathbb{N}$ such that for any $a, b \in S$, there is a K'-sequence from a to b of length at most m'.

Brandt Semigroups

Let $S = B^0(G, I)$ be a Brandt semigroup over a group G. Then S is pseudo-generated by a finite set X if and only if I is finite.

What can we say for semigroups?

Inverse semigroups

Let S be an inverse semigroup and $E(S)$ be the set of idempotents of S. Then S is pseudo-generated by a finite set (pseudo-finite) if and only if

1. there are finitely many maximal idempotents in $E(S)$ such that every idempotent is below a maximal idempotent;
2. $E(S)$ has a zero;
3. the group $\mathcal{H}$-class of zero is finitely generated (finite).

Thank You