SOME REMARKS ABOUT SEMIGROUPS OF PARTIAL CONTRACTION MAPPINGS OF A FINITE CHAIN

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NBSAN27 University of York.
7-8th January 2018

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1. Abstract

ABSTRACT. A general systematic study of the semigroups of partial contractions of a finite chain and their various subsemigroups of order-preserving/order-reversing and/or order-decreasing transformations was initiated in 2013 supported by a grant from The Research Council of Oman (TRC).

Our aim in this talk is to present the results obtained so far by the presenter and his co-authors as well as others. Broadly, speaking the results can be divided into two groups: algebraic and combinatorial enumeration. The algebraic results show that these semigroups are nonregular (left) abundant semigroups (for $n \geq 4$) whose set of idempotents forms a band. The combinatorial enumeration results show links with sequences some of which are in the encyclopedia of integers sequences (OEIS) and with others which are not.

2. Definitions and Notations

A transformation $\alpha \in \mathcal{P}_n$ is said to be

- *order-preserving (order-reversing)* if (for all $x, y \in \, Dom\,\alpha$) $x \leq y \implies x\alpha \leq y\alpha \; (x\alpha \geq y\alpha)$;
- *order-decreasing* (*order-increasing*) if (for all $x \in Dom\,\alpha$) $x\alpha \leq x \, (x\alpha \geq x)$.
- *a contraction* if (for all $x, y \in Dom\,\alpha$) $\mid x - y \mid \geq \mid x\alpha - y\alpha \mid$.

The semigroups of *order-preserving* transformations, *order-decreasing (extensive)* transformations, their intersections and their various generalizations are arguably the most studied subsemigroups of transformations.

Contractions	Full	Partial
Partial contractions	$\mathcal{CT}_n$	$\mathcal{CP}_n$
Order-preserving	$\mathcal{OCT}_n$	$\mathcal{OCP}_n$
Order-preserving or order-reversing	$\mathcal{ORCT}_n$	$\mathcal{ORCP}_n$
Order-decreasing	$\mathcal{DCT}_n$	$\mathcal{DCP}_n$
Order-preserving +order-decreasing	$\mathcal{ODCT}_n$	$\mathcal{ODCP}_n$
Order-reversing + order-decreasing	$\mathcal{ODCT}_n$	$\mathcal{DRCP}_n$

Table 1

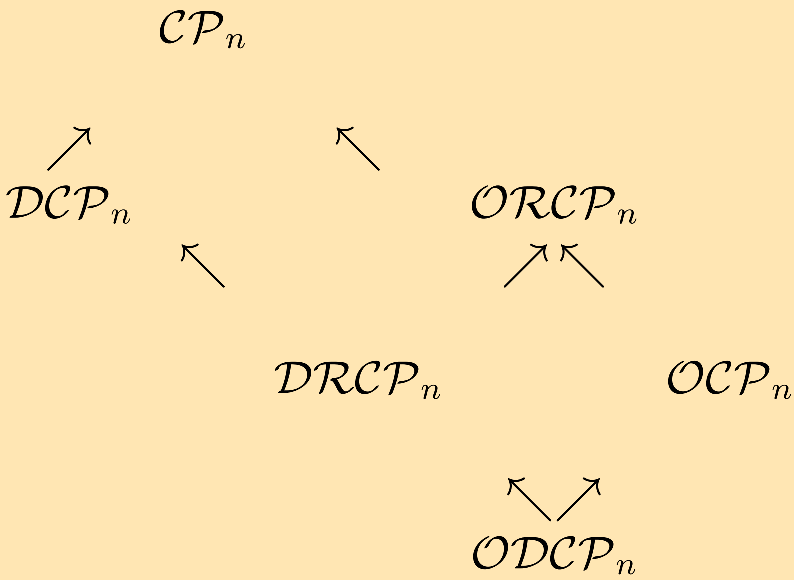
$\mathcal{CP}_n$
$\mathcal{DCP}_n$
$\mathcal{ORCP}_n$
$\mathcal{DRCP}_n$
$\mathcal{OCP}_n$
$\mathcal{ODCP}_n$

Figure 1

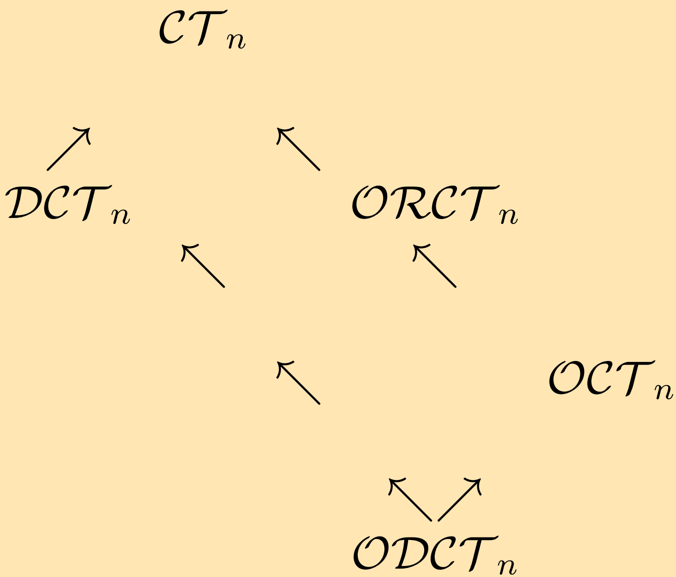
$\mathcal{CT}_n$
$\mathcal{DCT}_n$
$\mathcal{ORCT}_n$
$\mathcal{OCT}_n$
$\mathcal{ODCT}_n$

Figure 2

Let

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_p \\ a_1 & a_2 & \cdots & a_p \end{pmatrix} \in \mathcal{CP}_n,$$

with $a_i\alpha^{-1} = A_i$ (for $1 \leq i \leq p$).
A transversal $T_\alpha = \{t_i : t_i \in A_i\}$ of α is called *admissible* if

$$\begin{pmatrix} A_1 & A_2 & \cdots & A_p \\ t_1 & t_2 & \cdots & t_p \end{pmatrix} \in \mathcal{CP}_n.$$

A transversal $T_\alpha = \{t_i : t_i \in A_i\}$ of α is called *good* if $t_i \mapsto a_i$ is an isometry.

admissible transversal $\Leftarrow$ good transversal $\Leftarrow$ convex

Example 1 • $\begin{pmatrix} 1 & \{2,3\} & 4 \\ 1 & 2 & 3 \end{pmatrix} \in \mathcal{CP}_4$ *has no admissible transversal;*

• $\begin{pmatrix} 1 & \{2,4\} & 3 \\ 1 & 2 & 3 \end{pmatrix} \in \mathcal{CP}_4$ *has a convex transversal.*

3. Regularity and Green's Relations

Theorem 1 *Let $\alpha \in \mathcal{CP}_n$. Then α is regular iff α has a good transversal.*

Example 2

- $\begin{pmatrix} 1 & \{2,3\} & 4 \\ 1 & 2 & 3 \end{pmatrix} \in \mathcal{CP}_4$ *is not regular;*
- $\begin{pmatrix} 1 & \{2,4\} & 3 \\ 1 & 2 & 3 \end{pmatrix} \in \mathcal{CP}_4$ *is regular.*

Theorem 2 *Let* $\alpha, \beta \in \mathcal{CP}_n$. *Then* $(\alpha, \beta) \in \mathcal{R}$ *iff* $ker\alpha = ker\beta$ *and* $a_i \mapsto b_i$ *is an isometry.*

Example 3 *Consider*

- $\alpha = \begin{pmatrix} \{1,2\} & 3 & 5 \\ 1 & 2 & 3 \end{pmatrix}$,
  $\beta = \begin{pmatrix} \{1,2\} & 3 & 5 \\ 1 & 2 & 4 \end{pmatrix}, \gamma = \begin{pmatrix} \{1,2\} & 3 & 5 \\ 2 & 3 & 4 \end{pmatrix} \in \mathcal{CP}_5$.

Then $(\alpha, \beta) \notin \mathcal{R}$ *but* $(\alpha, \gamma) \in \mathcal{R}$.

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Theorem 3 *Let $\alpha, \beta \in \mathcal{CP}_n$. Then $(\alpha, \beta) \in \mathcal{L}$ iff (i) there exist admissible transversals T_α, T_β such that $t_i \mapsto t'_i$ is an isometry and $t_i\alpha = t'_i\beta$; or (ii) $A_i = B_i + e$ (for some integer e) and $A_i\alpha = B_i\beta$.*

Example 4 • $\begin{pmatrix} \{1,3\} & 2 & 5 \\ 1 & 2 & 3 \end{pmatrix} \mathcal{L} \begin{pmatrix} \{2,4\} & 3 & \{6,7\} \\ 1 & 2 & 3 \end{pmatrix}$,

but $\begin{pmatrix} \{1,3\} & 2 & 5 \\ 2 & 3 & 4 \end{pmatrix}$ *and* $\begin{pmatrix} \{2,4\} & 3 & \{6,7\} \\ 4 & 3 & 2 \end{pmatrix}$ *are not $\mathcal{L}$-related.*

Example 5 • $\begin{pmatrix} \{1,3\} & 2 & 5 \\ 1 & 2 & 3 \end{pmatrix} \mathcal{L} \begin{pmatrix} \{2,4\} & 3 & 6 \\ 1 & 2 & 3 \end{pmatrix}$, *but* $\begin{pmatrix} \{1,3\} & 2 & 5 \\ 2 & 3 & 4 \end{pmatrix}$ *and* $\begin{pmatrix} \{2,4\} & 3 & 6 \\ 4 & 3 & 2 \end{pmatrix}$ *are not* $\mathcal{L}$*-related.*

Theorem 4 *Let* $\alpha, \beta \in \mathcal{CP}_n$. *Then* $(\alpha, \beta) \in \mathcal{D}$ *iff (i) there exist admissible transversals* T_α, T_β *such that* $t_i \mapsto t_i'$ *and* $t_i\alpha \mapsto t_i'\beta$ *are isometries; or (ii)* $A_i = B_i + e$ *and* $A_i\alpha = B_i\beta + e'$ *(for some integers* e, e'*).*

Theorem 5 *Let $\alpha, \beta \in \mathcal{CP}_n$. Then we have the following:*

- $(\alpha, \beta) \in \mathcal{R}^*$ *iff* $ker\, \alpha = ker\, \beta$;
- $(\alpha, \beta) \in \mathcal{L}^*$ *iff* $\text{Im } \alpha = \text{Im } \beta$;
- $(\alpha, \beta) \in \mathcal{D}^*$ *iff* $\mid \text{Im } \alpha \mid = \mid \text{Im } \beta \mid$.

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Conjecture 1 *For $n \geq 3$, the semigroups $\mathcal{CT}_n$, $\mathcal{ORCT}_n$ and $\mathcal{OCT}_n$ are left abundant but not right abundant.*

Conjecture 2 *The sets $Reg(\mathcal{CT}_n)$, $Reg(\mathcal{ORCT}_n)$ and $Reg(\mathcal{OCT}_n)$ are semigroups.*

Conjecture 3 *The sets $E(\mathcal{CT}_n)$, $E(\mathcal{ORCT}_n)$ and $E(\mathcal{OCT}_n)$ are semigroups/bands.*

4. Combinatorial Results

It is now established that counting certain natural equivalence classes in various semigroups of partial transformations of an n-set, leads to very interesting enumeration problems. Many numbers and triangle of numbers regarded as combinatorial gems like the *Fibonacci number*, *Catalan number*, *Schröder number*, *Stirling numbers*, *Eulerian numbers*, *Narayana numbers*, *Lah numbers*, etc., have all featured in these enumeration problems.

- *breadth or width* of α: $b(\alpha) =\mid \operatorname{Dom}\alpha \mid$
- *height* or *rank* of α: $h(\alpha) =\mid \operatorname{Im}\alpha \mid$,
- *right [left] waist* of α:
  $w^{+}(\alpha) = \max(\operatorname{Im}\alpha)\ [w^{-}(\alpha) = \min(\operatorname{Im}\alpha)]$.
- *collapse* of α:
  $$c(\alpha) =\mid \bigcup\{t\alpha^{-1} : t \in \operatorname{Im}\alpha \textit{ and } \mid t\alpha^{-1} \mid \geq 2\} \mid,$$
- *fix* of α:
  $$f(\alpha) =\mid F(\alpha) \mid=\mid \{x \in \operatorname{Dom}\alpha : x\alpha = x\} \mid .$$

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Let S be a set of partial transformations on X_n. Next, let

$$F_{rqpmk}(n; r, q, p, m, k) \\ = \mid \{\alpha \in S : \wedge(b(\alpha) = r, c(\alpha) = q, h(\alpha) = p, \\ f(\alpha) = m, w^+(\alpha) = k)\} \mid$$

and, let $P = \{r, q, p, m, k\}$ be the set of counters for the breadth, collapse, height, fix and right waist of a transformation.

Then any 5-parameter combinatorial function can be expressed as $F(n; a_1, a_2, a_3, a_4)$, where $\{a_1, a_2, a_3, a_4\} \subset P$.

For example,

$$F_{rqpk}(n; r, q, p, k)$$
$$= \mid \{\alpha \in S : \wedge(b(\alpha) = r, c(\alpha) = q, h(\alpha) = p,$$
$$w^+(\alpha) = k)\} \mid .$$

	$\mathcal{T}_n$	$\mathcal{P}_n$
$F(n;r)$	n^n (if $r = n$) and 0 (if $r \neq n$)	$\binom{n}{r} n^r$
$F(n;q)$	?	?
$F(n;p)$	$\binom{n}{p} S(n,p)p!$	$\binom{n}{p} S(n+1,p+1)p!$
$F(n;m)$	$\binom{n}{m} (n-1)^{n-m}$	$\binom{n}{m} n^{n-m}$
$F(n;k)$	$k^n - (k-1)^n$	$(k+1)^n - k^n$

Table 2

		$\mathcal{P}_n$
$F(n;r,q)$		?
$F(n;r,p)$		$\binom{n}{r}\binom{n}{p}S(r,p)p!$
$F(n;r,m)$		$\binom{n}{m}\binom{n-m}{r-m}(n-1)^{r-m}$
$F(n;r,k)$		$\binom{n}{r}[k^r-(k-1)^r]$
$F(n;q,p)$		?
$F(n;p,k)$		$\binom{k-1}{p-1}S(n+1,p+1)p!$
$F(n;m,k)$		?

Table 3

S	$\mathcal{O}_n$	$\mathcal{PO}_n$
$\mid S \mid$	$\binom{2n-1}{n-1}$ [5]	$\sum_{r=0}^{n} \binom{n}{r}\binom{n+r-1}{r} = c_n$ [?, ?]
$\mid E(S) \mid$	F_{2n} [5]	e_n [?]
$\mid N(S) \mid$	0	?

Table 4

S	$\mathcal{O}_n$	$\mathcal{PO}_n$
$F(n;r)$	$\mid \mathcal{O}_n \mid$ or 0	$\binom{n}{r}\binom{n+r-1}{n-1}$ **[?]**
$F(n;q)$	?	?
$F(n;p)$	$\binom{n-1}{p-1}\binom{n}{p}$ **[?]**	$\binom{n}{p}e(n,p)*$ **[?]**
$F(n;m)$	$\frac{m}{n}\binom{2n}{n+m}$ **[?]**	?
$F(n;k)$	$\binom{n+k-2}{k-1}$ **[?]**	$\sum_{r=1}^{n}\binom{n}{r}\binom{k+r-2}{r-1}*$ **[?]**

Table 5

Theorem 6 *Let* $\alpha \in \mathcal{CP}_n$ *and let* A *be a convex subset of* $\mathrm{Dom}\,\alpha$*. Then* $A\alpha$ *is convex.*

Corollary 1 *Let* $\alpha \in \mathcal{CT}_n$*. Then* $\mathrm{Im}\,\alpha$ *is convex.*

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The results in Tables 6-7 were presented at MCCCC30 and will appear in the special issue of *JCMCC* dedicated to MCCCC30.

S	$\mathcal{ODCT}_n$	$\mathcal{OCT}_n$
$\mid S \mid$	2^{n-1} [1]	$(n+1)2^{n-2}$ [1]
$\mid E(S) \mid$	n [1]	$\binom{n+1}{2}$ [1]
$\mid N(S) \mid$	0	0

Table 6

S	$\mathcal{ODCT}_n$	$\mathcal{OCT}_n$
$F(n;r)$	$\mid \mathcal{ODCT}_n \mid$ or 0 [1]	$\mid \mathcal{OCT}_n \mid$ or 0 [1]
$F(n;q)$	?	?
$F(n;p)$	$\binom{n-1}{p-1}$ [1]	$(n-p+1)\binom{n-1}{p-1}$ [1]
$F(n;m)$	2^{n-m-1} [1]	$(n-m+3)2^{n-m-2}$ [1]
$F(n;k)$	$\binom{n-1}{k-1}$ [1]	$\sum_{p=1}^{k}\binom{n-1}{p-1}$ [1]

Table 7

We have the following results

Theorem 7 *Let* $S = \mathcal{ODCP}_n$. *Then* $f_m(x) = \sum_{n\geq 1} F(n;m)x^n = \left(\frac{x}{1-x}\right)^m \frac{x-2x^2}{B}$.

S	$\mathcal{ODCP}_n$	$\mathcal{OCP}_n$
$\mid S \mid$	$\frac{(2+\sqrt{2})^n+(2-\sqrt{2})^n}{2}$	$\frac{1-6x(1-x)^2}{B^2}$
$\mid E(S) \mid$	$1+n2^{n-1}$	$1+n(n+3)2^{n-3}$
$\mid N(S) \mid$	$\mid \mathcal{ODCP}_{n-1} \mid$	$1+\frac{x(1-x)(1-2x)^2}{B^2}$

Table 8

- $B = 1 - 4x + 2x^2$

- $\frac{1-6x(1-x)^2}{B^2}$
$= 1 + 2x + 8x^2 + 34x^3 + 140x^4 + 560x^5 + 2196x^6 + 8440x^7 + 32080x^8 + \cdots$
- $1 + \frac{x(1-x)(1-2x)^2}{B^2}$
$= 1 + x + 3x^2 + 12x^3 + 48x^4 + 188x^5 + 724x^6 + 2752x^7 + 10352x^8 + \cdots$

S	$\mathcal{ODCP}_n$	$\mathcal{OCP}_n$
$F(n;r)$	$\mid \mathcal{ODCP}_n \mid$ or 0	?
$F(n;q)$	?	?
$F(n;p)$	?	?
$F(n;m)$	$\left(\frac{x}{1-x}\right)^m \frac{x-2x^2}{B}$	$\frac{x^m(1-2x)^2}{(1-x)^{m-1}B^2}$
$F(n;k)$	?	?

5. Concluding Remarks

- If X_n is a POSET very little is known about these semigroups, both algebraically and combinatorially.
- The rank questions have not been investigated, except for $\mathcal{OCI}_n$.
- Products of nilpotents have not been investigated.
- Congruences have not been investigated.

Best wishes to Laszlo Marki on the occassion of your 70th birthday.

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