Extensions and I-semidirect products of inverse semigroups

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Introduction: groups and semigroups

extension of groups

semidirect / standard wreath product of groups

Theorem (Kaloujnine–Krasner)

Every extension of a group K by a group T is embeddable in the standard wreath product of K by T.

If G is a group and K is a normal subgroup in G, i.e., G is an extension of K by $T \stackrel{\text{def}}{=} G/K$, then there is an embedding $\iota\colon G \to K \,\mathrm{Wr}\, T$ such that $\iota(g) = (\ \ , gK)$ for every $g \in G$.

'extension' of semigroups cannot be defined in general

semidirect / standard wreath product of semigroups are defined,
and play significant role in structure theory from the 1950's:

Krohn–Rhodes theorem

division instead of embedding

Introduction: inverse semigroups

extension of inverse semigroups can be introduced:

— the idempotent θ-classes determine the congruence θ

— $\operatorname{Ker}\theta \stackrel{\text{def}}{=}$ union of the idempotent congruence classes, a semilattice of inverse subsemigroups

S, K, T inverse semigroups

Definition (extension of inverse semigroups)

We say that S is an extension of K by T if there exists a congruence θ on S s.t. $\operatorname{Ker}\theta$ is isomorphic to K and S/θ to T. If for any θ-class C, the set $\{c^{-1}c : c \in C\}$ has a maximum element, then we call S a *Billhardt extension of K by T*.

Example: each idempotent separating extension is Billhardt, but this fails for idempotent pure extensions

Introduction: inverse semigroups

semidirect / wreath product of inverse semigroups
need not be inverse

$\Longrightarrow$ modified constructions are introduced:

— wreath product taylored for inv.sgs (Houghton, 1976)

— λ-semidirect / λ-wreath product (Billhardt, 1992)

$K *^\lambda T$ — extension of K' by T, where K' is an inverse subsemigroup of $K \times E(T)$

$K\ \mathrm{Wr}^\lambda\ T$ — extension of K' by T, where K' is an inverse subsemigroup of $K^T \times E(T)$

$K\ \mathrm{Wr}^H\ T$ — extension of K' by T, where K' is a strong semilattice of direct powers of K by T

Introduction: inverse semigroups

Theorem (Houghton)

Each idempotent separating extension of K by T is embeddable in Houghton's wreath product of K by T.

Theorem (Billhardt)

Every Billhardt extension of K by T is embeddable in the λ-wreath product of K by T.

Theorem (Billhardt)

Each idempotent pure extension by T is embeddable in a λ-semidirect product of a semilattice by T.

A new product of inverse semigroups

Motivation:
criticism of M. Kambites (2015) on λ-semidirect product

We remark that the λ-semidirect product is somewhat unusual, indeed arguably even unnatural, in the context of inverse semigroup theory. Inverse semigroups usually arise as models of "partial symmetry", and by the Wagner–Preston Theorem can all be represented as such. It is thus customary (and almost always most natural) to consider them acting by partial bijections, rather than by functions, and an action by endomorphisms thus seems intuitively like a "category error".

The same criticism is valid for Houghton's wreath product.

A new product of inverse semigroups

Construction

K, T — inverse semigroups

$K = \bigcup_{e \in E(T)} K_e$, where K_e is an inverse subsemigroup in K, and $K_e K_f \subseteq K_{ef}$ for every $e, f \in E(T)$

T acts on K by partial automorphisms such that

$$t \circ K_e \subseteq K_{tet^{-1}} \quad \text{for every} \quad e \leq t^{-1}t$$

Define

$$K \rtimes^{l} T \stackrel{\text{def}}{=} \{(a, t) \in K \times T : a \in K_{tt^{-1}}\}$$

with multiplication

$$(a, t)(b, u) \stackrel{\text{def}}{=} \left(t \circ ((t^{-1} \circ a)b), tu\right).$$

I-semidirect product of inverse semigroups

Proposition

$K \rtimes^I T$ is an inverse semigroup which is an extension of K' by T, where K' is a semilattice $E(T)$ of K_e ($e \in E(T)$).

Definition

We call $K \rtimes^I T$ an *I-semidirect product of K by T*.

Relation to former products (NSAC 2017)

- each of $K *^\lambda T$, $K \text{ Wr}^\lambda T$ and $K \text{ Wr}^H T$ is isomorphic to an I-semidirect product of K' by T
- a subclass of I-semidirect products is found which is equivalent to any of the former classes of products from the point of view of which extensions of inverse semigroups are embeddable in them

Extensions of inverse semigroups

Theorem

Every extension of inverse semigroups is embeddable in an I-semidirect product.

More precisely, if S is any inverse semigroup and θ is any congruence on S, i.e., S is an extension of $K \stackrel{\text{def}}{=} \operatorname{Ker}\theta$ by $T \stackrel{\text{def}}{=} S/\theta$, then there exists an inverse semigroup $\overline{K}$, an I-semidirect product $\overline{K} \rtimes^{I} T$ and an embedding $\iota\colon S \to \overline{K} \rtimes^{I} T$ such that $\iota(a) = (\quad, a\theta)$ for every $a \in S$.

Remark

Recall that, in the former embedding theorems, the first factor of the product was ‘close to’ the kernel of the extension (e.g., they belonged to the same variety).

This is not the case with $\overline{K}$ found in the proof of our theorem.

Extensions of inverse semigroups

S — inverse semigroup
θ — congruence on S
$T \stackrel{\text{def}}{=} S/\theta$

$\mathcal{IS} = \mathcal{IS}(S, \theta)$ — 'ordered' inverse semigroupoid, where:

$\operatorname{Obj} \mathcal{IS} \stackrel{\text{def}}{=} T$

$\mathcal{IS}(u, v) \stackrel{\text{def}}{=} \{(u, a, v) \in T \times S \times T : uu^{-1} = vv^{-1},\ a \in u^{-1}v\}$

$\exists (u, a, v)(x, b, y) \Leftrightarrow v = x$, and if this is the case, then

$$(u, a, v)(v, b, y) \stackrel{\text{def}}{=} (u, ab, y)$$

$(u, a, v) \leq (x, b, y) \Leftrightarrow u \leq x,\ a \leq b,\ v \leq y$

pseudoproduct can be defined:

$\exists (u, a, v) \otimes (x, b, y) \Leftrightarrow v \sim_\ell x$, and if this is the case, then

$$(u, a, v) \otimes (x, b, y) \stackrel{\text{def}}{=} (vx^{-1}u, ab, vx^{-1}y)$$

Extensions of inverse semigroups

left translations induce an action of T on $\mathcal{IS}$ by partial automorphisms:

$$\exists t \circ u \quad \text{and} \quad \exists t \circ (u, a, v) \quad \Longleftrightarrow \quad t^{-1}t \geq uu^{-1} (= vv^{-1})$$

and if this is the case, then

$$t \circ u \overset{\text{def}}{=} tu \quad \text{and} \quad t \circ (u, a, v) \overset{\text{def}}{=} (tu, a, tv)$$

the action respects the pseudoporduct:

$$\exists t \circ (u, a, v), \quad \exists t \circ (x, b, y) \quad \text{and} \quad \exists (u, a, v) \otimes (x, b, y)$$
$$\Downarrow$$
$$\exists t \circ \big((u, a, v) \otimes (x, b, y)\big), \quad \exists \big(t \circ (u, a, v)\big) \otimes \big(t \circ (x, b, y)\big)$$
$$\text{and}$$
$$t \circ \big((u, a, v) \otimes (x, b, y)\big) = \big(t \circ (u, a, v)\big) \otimes \big(t \circ (x, b, y)\big)$$

Extensions of inverse semigroups

V — a variety of inverse semigroups
I — the variety of (all) inverse semigroups
G — the variety of groups

$F\mathbf{V}_A$ — the free object in **V** on the set A

$A \stackrel{\text{def}}{=} \text{Arr}\,\mathcal{IS}$
$\rho_{\mathbf{V}}$ — least congruence on $F\mathbf{V}_A$ identifying ab and $a \otimes b$
for any $a, b \in A$ s.t. $\exists a \otimes b$

the action of T on A extends to an action on $F\mathbf{V}_A/\rho_{\mathbf{V}}$ by partial automorphisms $\implies$ $(F\mathbf{V}_A/\rho_{\mathbf{V}})\,\text{Wr}^I\, T$ is defined

main step o the proof: for every $a, b \in S$,

$$\big((aa^{-1})\theta, a, a\theta\big)\ \rho_{\mathbf{I}}\ \big((bb^{-1})\theta, b, b\theta\big) \quad \implies \quad a = b$$

consequently,

$$S \to (F\mathbf{V}_A/\rho_{\mathbf{V}})\,\mathrm{Wr}^I\, T, \quad a \mapsto \Big(\big((aa^{-1})\theta, a, a\theta\big)\rho_{\mathbf{I}}, a\theta\Big)$$

is an embedding

Similar argument for group extensions, with **I** replaced by **G**:
$F\mathbf{G}_A/\rho_{\mathbf{G}}$ is far from a subgroup of K^T