Cross-connection of the Inductive Groupoid of a Regular Semigroup

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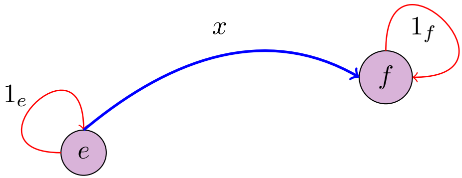
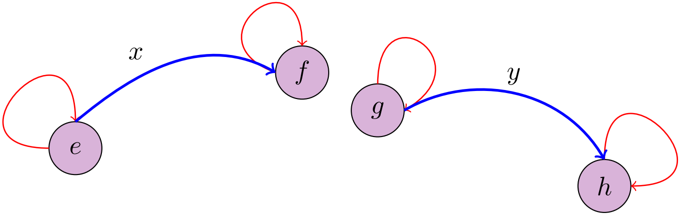
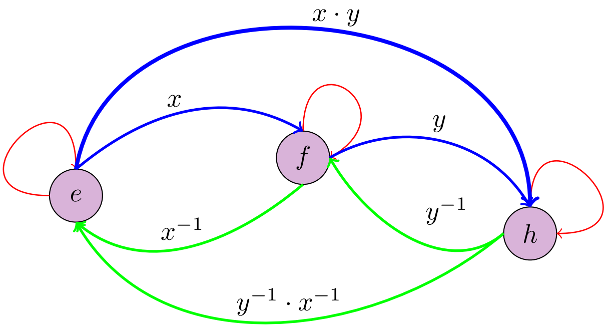
named after the first President of Russia B.N.Yeltsin

- In structure theory of semigroups, Ehresmann (1957) and Schein (1965) used categories to describe inverse semigroups.
- Their approach relied on the fact that a groupoid can be naturally associated with an inverse semigroup.
- Recall that every element x in an inverse semigroup has a unique inverse element x^{-1}. Then xx^{-1} and $x^{-1}x$ are idempotents.
- So, given an element x in an inverse semigroup, it can be seen as an arrow (or a morphism) from $xx^{-1} = e$ to $x^{-1}x = f$.





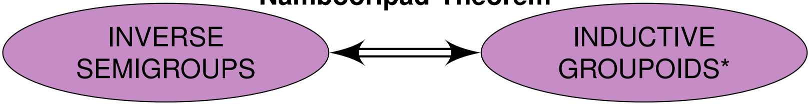
- Suppose $f = g$, that is if $xx^{-1} = y^{-1}y$, then a partial (associative) composition can be defined by restricting the binary composition of the inverse semigroup.



- When $xx^{-1} = y^{-1}y = f$, then a partial (associative) composition is defined as above.
- This forms a category with idempotents forming the set of vertices (objects).
- Also, every morphism in this category has an inverse morphism.

- Thus a groupoid $\mathcal{G}(S)$ is associated with an inverse semigroup S.
- The idempotents of an inverse semigroup under the natural partial order form a semilattice.
- So, the vertex set $v\mathcal{G}(S)$ of the groupoid $\mathcal{G}(S)$ is a semilattice.
- Further, the morphisms in $\mathcal{G}(S)$ respect the order structure of the vertices. That is, each arrow can be restricted (corestricted) to idempotents below its domain (range).
- Such an associated groupoid is called an inductive groupoid*.
- Conversely, given an axiomatically defined inductive groupoid*, there is a unique associated inverse semigroup.

Ehresmann-Schein-Nambooripad Theorem



Definition

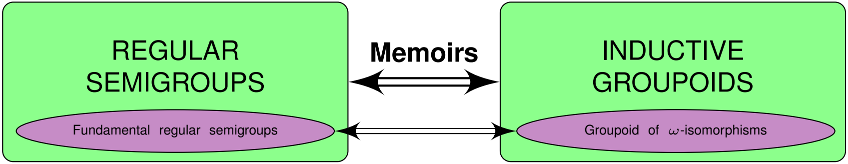
A semigroup S is said to be (von Neumann) regular if every element has at least one inverse element.

- In AMS Memoirs No. 224 (1979), Nambooripad extended the ESN correspondence to regular semigroups.
- This was based on his Ph. D. thesis (1973).
- His initial approach (1975) was using structure mappings instead of restrictions/corestrictions.
- Structure mappings approach was independently explored by Meakin (1976).
- In Memoirs, the order structure of idempotents of a (regular) semigroup was axiomatised as a (regular) biordered set.
- Further, an additional layer of biorder structure using a new groupoid of E-chains was added to the inductive groupoid.

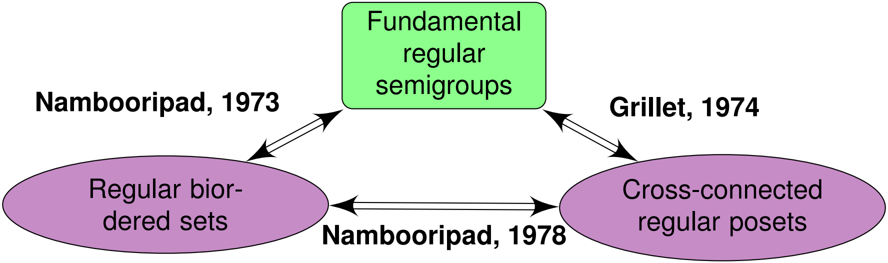
- So, Nambooripad could recover the regular semigroup from his inductive groupoid.
- Thus he proved the equivalence between the categories of regular semigroups and inductive groupoids.
- Further, he specialised his results to generalise Munn's beautiful structure theorem (1970) for fundamental inverse semigroups.

Definition

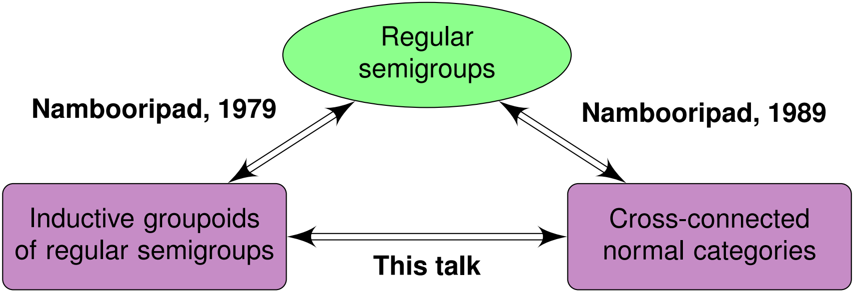
A regular semigroup is said to be fundamental if it cannot be homomorphically shrunk without collapsing its idempotents.



- Meanwhile, Grillet (1974) also constructed fundamental regular semigroups using the (regular) posets of its principal ideals.
- For this, he introduced the notion of cross-connections to describe the relationship between the posets.
- In 1978, Nambooripad showed that a regular biordered set is equivalent to a cross-connected pair of regular posets.



- Observe that a poset can be viewed a strict preorder category.
- Elaborating this idea, Nambooripad (1989) constructed arbitrary regular semigroups from cross-connected normal categories.



- We shall establish a direct equivalence between these two different structural invariants of the regular semigroup.

"The Telegraph is Like a Very, Very Long Cat" - Einstein (?)

"You see, wire telegraph is a kind of a very, very long cat. You pull his tail in New York and his head is meowing in Los Angeles. The radio operates exactly the same way : you send signals here, they receive them there. The only difference is that there is no cat."

- Our equivalence is like this. It works without the regular semigroup in between !
- In the process, we shall see how differently regular semigroups are encoded into these different abstract structures.
- Also, the discussion shall clarify the relationship between the ideal structure and the idempotent structure of a regular semigroup.
- We begin with the outline of the easier side.

Cross-connections ⇒ Inductive Groupoids

It can be shown that the 'cross-connected pair of isomorphisms' of the normal categories form an inductive groupoid.
This groupoid is inductive isomorphic to the inductive groupoid of the given semigroup.

Inductive Groupoids ⇒ Cross-connections

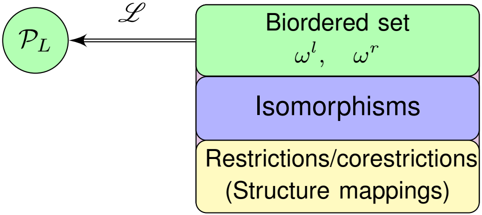
Recall that the set of objects of the inductive groupoid is a regular biordered set with quasi-orders, say ω^l and ω^r. Then let

$$\mathscr{L} = \omega^l \cap (\omega^l)^{-1} \text{ and } \mathscr{R} = \omega^r \cap (\omega^r)^{-1}.$$

We need to build two normal categories, say $\mathcal{L}_G$ and $\mathcal{R}_G$ such that they are cross-connected. To build the category $\mathcal{L}_G$, we first build three intermediary categories $\mathcal{P}_L$, $\mathcal{G}_L$ and $\mathcal{Q}_L$ such that

$$v\mathcal{P}_L = v\mathcal{G}_L = v\mathcal{Q}_L = v\mathcal{L}_G = E/\mathscr{L}.$$

INDUCTIVE GROUPOID



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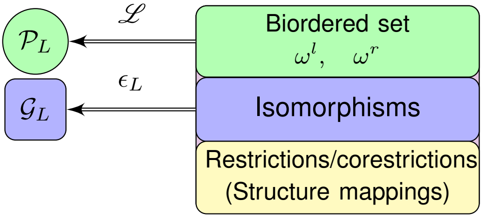

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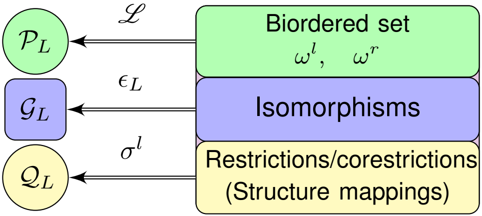

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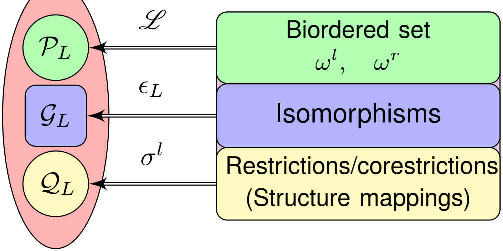

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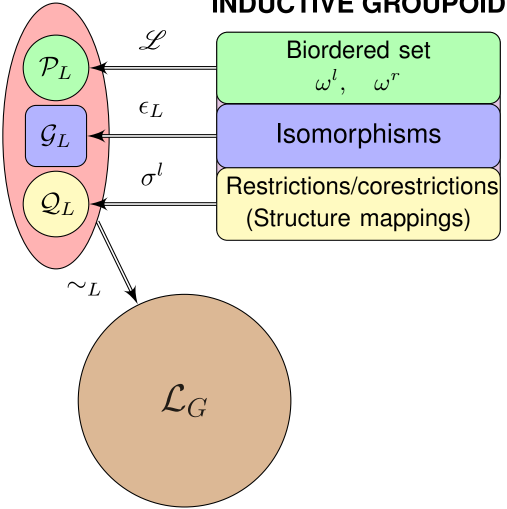
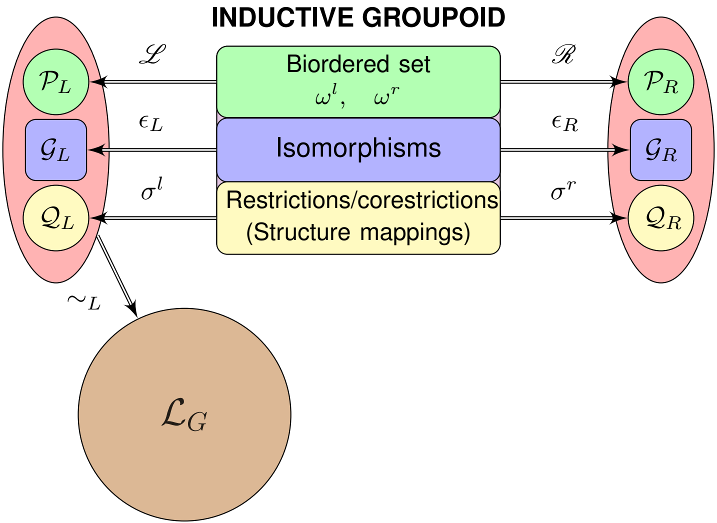
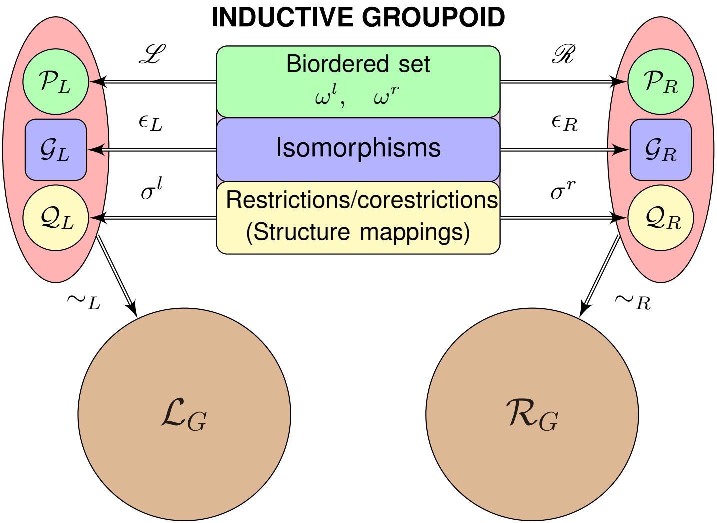



INDUCTIVE GROUPOID
Biordered set
$\omega^l, \quad \omega^r$
Isomorphisms
Restrictions/corestrictions
(Structure mappings)
$\mathscr{L}$
$\mathscr{R}$
ϵ_L
ϵ_R
σ^l
σ^r
$\mathcal{P}_L$
$\mathcal{G}_L$
$\mathcal{Q}_L$
$\mathcal{P}_R$
$\mathcal{G}_R$
$\mathcal{Q}_R$
$\sim_L$
$\mathcal{L}_G$



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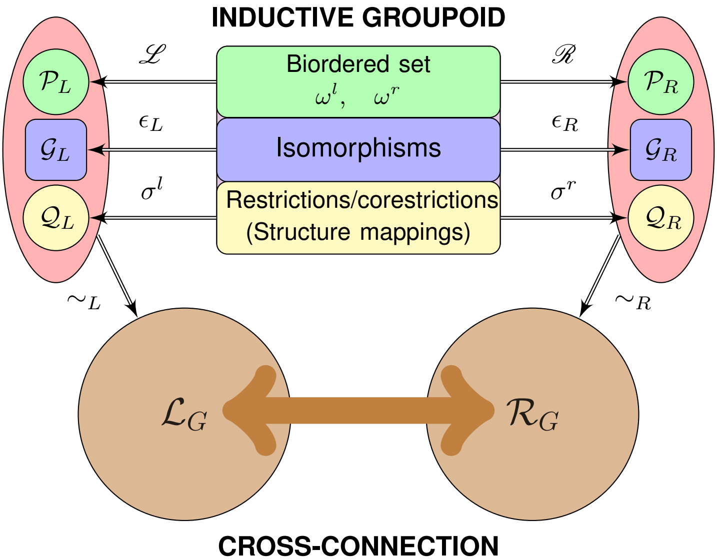
$\mathcal{L}_G$
$\mathcal{R}_G$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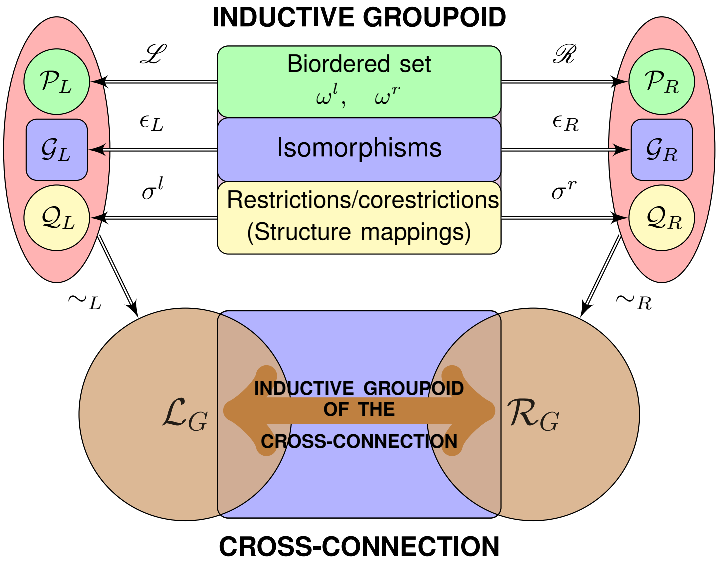
CROSS-CONNECTION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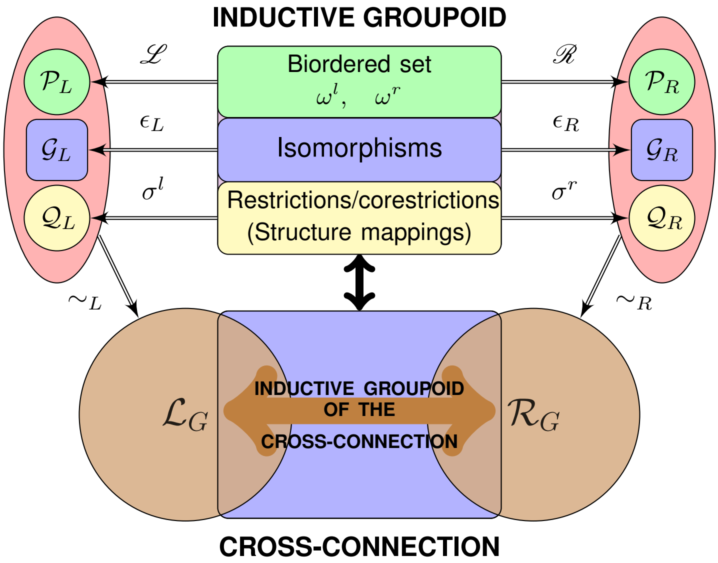
$\sim_L$
$\sim_R$
$\mathcal{L}_G$
INDUCTIVE GROUPOID
OF THE
CROSS-CONNECTION
$\mathcal{R}_G$
CROSS-CONNECTION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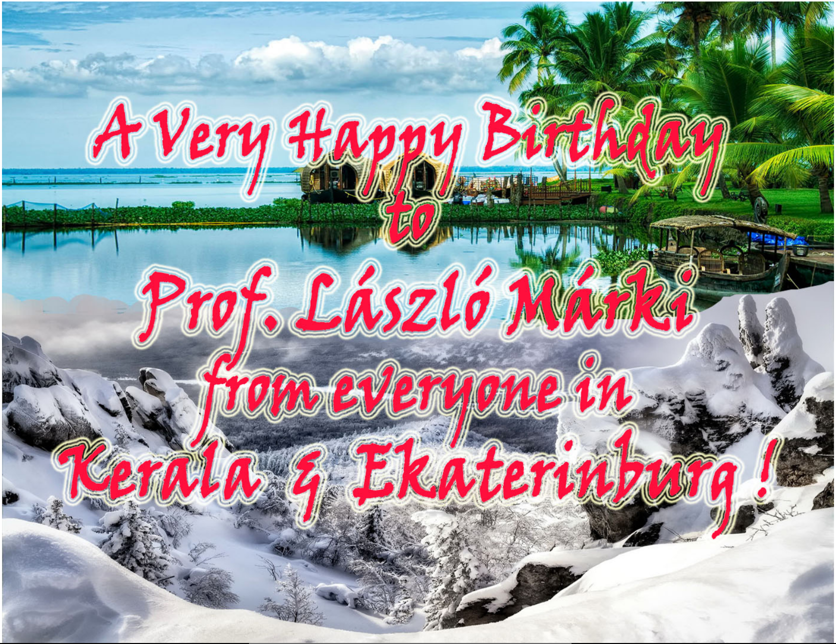
A Very Happy Birthday
to
Prof. László Márki
from everyone in
Kerala & Ekaterinburg !