Coherency and purity for monoids

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What's in this talk

1. What is coherency and why is it interesting?
2. S-acts over a monoid S
3. Which monoids are coherent?
4. Coherency and constructions
5. Coherency and purity
6. Questions

Throughout, S will denote a monoid.

Finitary conditions

Finitary condition (for a monoid)

A condition satisfied by all finite monoids.

Example

Every element of S has an idempotent power.

Finitary conditions were introduced by **Noether** and **Artin** in the early 20th Century to study rings; they changed the course of algebra entirely.

Example

Every right congruence on S is finitely generated, i.e. S is **right Noetherian**.
Every right ideal of S is finitely generated, i.e. S is **weakly right Noetherian**.

Coherency for Monoids: the definition

Coherency

This is the main finitary condition of importance to us today

Definition

S is right coherent if every finitely generated S-subact of every finitely presented right S-act is finitely presented.[a]

[a] This definition comes from Wheeler (1976)

Left coherency is defined dually: S is coherent if it is left and right coherent.

1. What is coherency and why is it interesting?
Why is coherency interesting?

- The definition is natural, and fits with that for rings.
- Coherency is defined in terms of S-acts, and understanding monoids in terms of their acts is surely important.
- It has connections with the model theory of S-acts.
- A 'Chase type' condition involving (right) annihilator congruences exists (**G**).
- Certain nice classes of monoids are coherent (more later).
- It has connections with products and ultraproducts of flat left S-acts (**Bulman-Fleming and McDowell, G, Sedaghatjoo**).
- Coherency is related to **purity** (more later).

2. S-acts over a monoid S
Representation of monoid S by mappings of sets

A **(right)** S**-act** is a set A together with a map

$$A \times S \to A,\ (a, s) \mapsto as$$

such that for all $a \in A, s, t \in S$

$$a1 = a \text{ and } (as)t = a(st).$$

Beware: an S-act is also called an S-set, S-system, S-action, S-operand, or S-polygon.

Let Act-S denote the class of all S-acts.

2. S-acts over a monoid S
Standard definitions/Elementary observations

- S-acts form a variety of universal algebras, to which we may apply the usual notions of subalgebra (S-**subact**), congruence, morphism (S-**morphism**), factor/quotient S-act, finitely generated, etc.
- S-acts and S-morphisms form a category, **Act-S**.
- We have usual definitions of **free, projective, injective**, etc. including variations on **flat**.

2. S-acts over a monoid S
Standard definitions/Elementary observations

- Any **right ideal** is a (right) S-act.
- S itself is a (right) S-act, the **free monogenic S-act**.
- Free S-acts are **disjoint unions of copies of S**.
- A is **finitely presented** if

$$A \cong F_S(X)/\rho$$

for some finitely generated free S-act $F_S(X)$ and finitely generated congruence ρ. If $|X| = 1$, then ρ is just a **right congruence** on S.

3. Which monoids are right coherent?
First observations

Definition

S is right coherent if every finitely generated S-subact of every finitely presented (right) S-act is finitely presented.

Theorem: Normak (77)

If S is right noetherian then S is right coherent.

Example: Fountain (92)

There is a monoid S which is weakly right noetherian but which is not right coherent.

Let us call the monoid in the example above the **Fountain monoid**: it is constructed from a group and a 4 element nilpotent semigroup.

3. Which monoids are right coherent?
20th century

Theorem: G (1992)

The following monoids are right coherent:

- groups;
- semilattices;
- Clifford monoids;
- regular monoids for which every right ideal is principal;
- the free commutative monoid on X.

3. Which monoids are right coherent?
21st century

Theorem: G, Hartmann, Ruškuc (2015)

Any free monoid X^* is coherent.

Theorem: K.G. Choo, K.Y. Lam and E. Luft (1972)

Free rings are coherent.

Theorem: G, Hartmann (2016)

Free inverse monoids are NOT right (or left) coherent.
Free left restriction monoids are right (but not left) coherent.

Many questions remain.....

Question

Is $\mathcal{I}_X$ coherent?

4. Coherency and constructions

G, Hartmann, Ruškuc - coming some day soon..

The class of right coherent monoids is ....

1. not closed under morphisms;
2. closed under retracts;
3. not closed under direct product;
4. closed under direct products with any finite monoid;
5. closed under taking certain monoid subsemigroups e.g. $\mathcal{J}$-classes or some $\widetilde{\mathcal{H}}_E$-classes.

4. Coherency and constructions
G, Hartmann, Ruškuc - coming some day soon..

Let M be a monoid and $I \neq \emptyset$. Then $\mathcal{B}^0(M; I)$ is defined by

$$\mathcal{B}^0(M; I) = I \times M \times I \cup \{0\}$$

with multiplication of non-zero elements given by

$$(i, m, j)(k, n, \ell) = \begin{cases} (i, mn, \ell) & \text{if } j = k \\ 0 & \text{else} \end{cases}$$

Theorem

Let M be a monoid. Then $\mathcal{B}^0(M; I)^1$ is right coherent if and only if M is right coherent.

5. Coherency and purity
Equations and inequations over S-acts

Let A be an S-act. An **equation** over A has the form

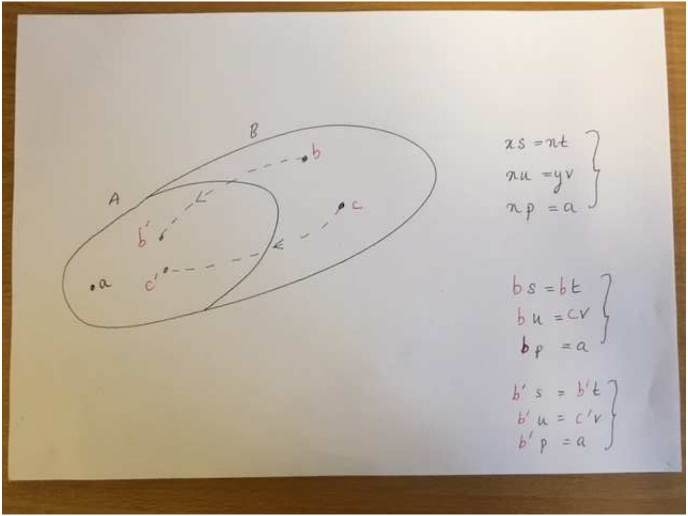
$$xs = xt,\ xs = yt \text{ or } xs = a$$

where x, y are variables, $s, t \in S$ and $a \in A$.

Consistency

A set of equations is **consistent** if it has a solution in some S-act $B \supseteq A$.

5. Coherency and purity
Equations and inequations over S-acts



Absolutely pure and almost pure

A is **absolutely pure (almost pure)** if every finite consistent set of equations over A (in 1 variable) has a solution in A.

absolutely pure	=	algebraically closed
almost pure	=	1-algebraically closed

5. Coherency and purity
Equations and inequations over S-acts

Let A be an S-act. An **inequation** over A has the form

$$xs \neq xt,\ xs \neq yt \text{ or } xs \neq a$$

where x, y are variables, $s, t \in S$ and $a \in A$.

Consistency

A set of equations and inequations is **consistent** if it has a solution in some S-act $B \supseteq A$.

Existentially closed and 1-existentially closed

A is **existentially closed (1-existentially closed)** if every finite consistent set of equations and inequations over A (in 1 variable) has a solution in A.

Theorem: Wheeler (1976); G (1986), Ivanov (1992)

The following are equivalent for a monoid S:

1. S is right coherent;
2. the existentially closed S-acts $\mathcal{E}$ form an axiomatisable class;
3. the first-order theory of S-acts has a model companion;
4. the 1-existentially closed S-acts $\mathcal{E}_1$ form an axiomatisable class;
5. the absolutely pure S-acts $\mathcal{A}$ form an axiomatisable class;
6. the almost pure S-acts $\mathcal{A}_1$ form an axiomatisable class.

5. Coherency and purity
Completely right pure monoids

Definition: A monoid is completely right pure

if every S-act is absolutely pure.

Clearly

$$\mathcal{A} \subseteq \mathcal{A}_1 \subseteq \text{Act}\text{-}S.$$

Theorem: G (1991)

A monoid S is completely right pure if and only if all S-acts are almost pure, i.e.

$$\text{Act}\text{-}S = \mathcal{A}_1 \Longleftrightarrow \text{Act}\text{-}S = \mathcal{A}.$$

5. Coherency and purity
Completely right pure monoids

Absolute purity for an S-act A is a form of weak injectivity. Thus, **completely right injective monoids** are **completely right pure**.

Completely right injective monoids were characterised by **Skornjakov (1979)** (using right ideals and right congruences), and **Fountain (1974)** and **Isbell (1972)** (using right ideals and elements).

The fact Act-$S = \mathcal{A}_1 \iff$ Act-$S = \mathcal{A}$ enabled me to characterise **completely right pure monoids (1991)** in analogous ways.

A Question

Does there exist a monoid S and an S-act A such that A is almost pure but not absolutely pure? In other words, do we **always** have $\mathcal{A} = \mathcal{A}_1$????

5. Coherency and purity
The Question: does $\mathcal{A} = \mathcal{A}_1$ for every monoid S?

Theorem: G, Yang Dandan, Salma Shaheen (2016)

Let S be a finite monoid. Then $\mathcal{A} = \mathcal{A}_1$.

Consequently: $\mathcal{A} = \mathcal{A}_1$ is a finitary condition.

Theorem: G, Yang Dandan (2016/7)

Let S be a right coherent monoid. Then $\mathcal{A} = \mathcal{A}_1$.

5. Coherency and purity
Absolute purity vs almost purity
New Question: does $\mathcal{A} = \mathcal{A}_1$ if and only if S is right coherent?

Answer: No!

Example G, Yang Dandan (2017)

The Fountain Monoid is an example of a non-coherent monoid such that $\mathcal{A} = \mathcal{A}_1$.

5. Coherency and purity
The Question: does $\mathcal{A} = \mathcal{A}_1$ for every monoid S?

For an S-act A we can build canonical absolutely pure (almost pure) extensions $A(\aleph_0)$ ($A(1)$).

Proposition G: 2017

The following are equivalent for a monoid S:

1. every almost pure S-act is absolutely pure;
2. for every **finitely generated subact of every finitely presented S-act** A, we have $A(1)$ is a retract of $A(\aleph_0)$.

6. Questions, Questions...!

- Does there exist a monoid S and an S-act A such that A is almost pure but not absolutely pure? Use the result that $A(1)$ is always a retract of $A(\aleph_0)$ to write down a condition on chains of right congruences such that every almost pure S-act is absolutely pure; **now find an S not satisfying this condition.**
- What happens for rings and modules? Are the almost pures (1-algebraically closed) absolutely pure (algebraically closed)?
- When is $\mathcal{E} = \mathcal{E}_1$?
- Is $\mathcal{I}_X$ coherent? Is $\mathcal{T}_X$ right (or left) coherent?
- Coherency and constructions: e.g. for which M, P is $\mathcal{M}^0(M; I, \Lambda; P)^1$ right coherent?
- Determine exact connections of right coherency with products/ultraproducts of flat **left** S-acts.
- Other finitary conditions arise from model theoretic considerations of S-acts; many open questions remain!

One final thing

Many more happy years of mathematics for Laci!