Subdirect products of free semigroups

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Motivation and prior work

Definition

Let A and B be algebras of the same type. A subalgebra C of the direct product $A \times B$ is called a **subdirect product** of A and B if the canonical projection maps

$$\pi_A : C \to A := (a, b) \mapsto a$$

$$\pi_B : C \to B := (a, b) \mapsto b$$

are surjections. We write $C \leq_{s.d} A \times B$ to denote this.

Note that the direct product itself is an example of a subdirect product.

Motivation and prior work

Subdirect products of groups (and in particular, direct products) have been well studied, and many results of the form

$$\text{“}A \times B \text{ has property } \mathcal{P} \iff A \text{ and } B \text{ have property } \mathcal{P}\text{“.}$$

hold for the direct product (including the finiteness properties, solvability, nilpotency, word problem decidability, etc.), but this situation can radically change for subgroups of the direct product (and in particular, for subdirect products).

A, B	P	True?	Ref
Finitely generated groups	“has countably many subgroups“	✗	Baumslag -Roseblade (1984)
(Infinite) Semigroups	Finitely generated (iff $A^2 = A, B^2 = B$)	✓	(R,R,W 1998)
	Finitely presented (iff stable)	✓	“
	Residually finite	✓	Gray, Ruškuc 2009
(Infinite) Lattices	Finitely generated	✓	Mayr, Ruškuc 2016
	Finitely presented	✗	“
Rings	Finitely generated	✓	“
	Finitely presented	✓	“

Table: Some particular examples

$$\text{“}A \times B \text{ has property } \mathcal{P} \iff A \text{ and } B \text{ have property } \mathcal{P}\text{“.}$$

Motivation and prior work

We want to extend this study to direct and subdirect products of semigroups and monoids. We begin with some Baumslag-Roseblade type counterexamples involving the free monogenic semigroup $\mathbb{N}$:

Theorem A

There are uncountably many non-isomorphic subsemigroups of $\mathbb{N} \times \mathbb{N}$.

Theorem B

For any $k \geq 2$, the direct power $\mathbb{N}^k$ contains uncountably many non-isomorphic subdirect products.

Motivation and prior work

And, as a corollary of Theorem A:

Corollary C

If S, T are infinite semigroups containing elements of infinite order, then $S \times T$ contains uncountably many non-isomorphic subsemigroups.

We consider subsemigroups of the form

$$S_M := \langle (1, m) : m \in M \rangle$$

for $M \subseteq \mathbb{N}$. Note that if $|M| \leq 2$, then S_M is isomorphic to a free semigroup.

Lemma 1

Let $M = \{m_1, m_2, m_3\}$, $N = \{n_1, n_2, n_3\}$ be two 3-element subsets of $\mathbb{N}$. Then S_M and S_N are isomorphic, via isomorphism $\varphi : S_M \to S_N$ satisfying $(1, m_i)\varphi = (1, n_i)$ $(i = 1, 2, 3)$, if and only if

$$n_2(m_3 - m_1) = n_1(m_3 - m_2) + n_3(m_2 - m_1).$$

We will define a subset $M \subseteq \mathbb{N}$ (with $|M| \geq 3$) to be 3-*separating* if;

(S1) For any two triples (m_1, m_2, m_3) and (n_1, n_2, n_3) of distinct elements from M,

$$\begin{aligned} & n_2(m_3 - m_1) = n_1(m_3 - m_2) + n_3(m_2 - m_1) \\ \iff \quad & (m_1, m_2, m_3) = (n_1, n_2, n_3). \end{aligned}$$

For example, the set $M = \{1, 2, 3\}$ is not 3-separating, as the pairs $(1, 2, 3)$ and $(3, 2, 1)$ violate condition (S1), but the set $N = \{1, 2, 4\}$ is. Further, we will say a set is **strongly 3-separating** if it is 3-separating, and

(S2) For any two pairs (m_1, m_2), (n_1, n_2) of distinct elements of M:

$$m_1 - m_2 + n_2 - n_1 = 0 \iff (m_1, m_2) = (n_1, n_2).$$

Lemma 2

If M is a strongly 3-separating finite set, then there exists $x \in \mathbb{N} \setminus M$ such that $M \cup \{x\}$ is also strongly 3-separating.

Corollary 1

There exists an infinite 3-separating set M_∞.

Proof: Choose a strongly 3-separating set M_1 which is finite for a starting point. Let $M_{i+1} = M_i \cup \{x_i\}$ for $i \in \mathbb{N}$, where $x_i \in \mathbb{N} \setminus M_i$ is the least natural number such that $M_i \cup \{x_i\}$ is strongly 3-separating. Let $M_\infty = \bigcup_{i \in \mathbb{N}} M_i$.

Theorem A

There are uncountably many non-isomorphic subsemigroups of $\mathbb{N} \times \mathbb{N}$.

Proof: Take the collection of subsemigroups defined

$$\mathcal{C} = \{S_M : M \subseteq M_\infty\}.$$

Suppose that two of these semigroups S_M and S_N were isomorphic, and let φ be an isomorphism between them. If $M \neq N$, we can assume w.l.o.g that $M \setminus N \neq \emptyset$. Then in particular there would exists $m_1, m_2, m_3 \in M$ distinct, $n_1, n_2, n_3 \in N$ also distinct such that

1. At least one of the m_i is not equal to any of the n_j,
2. $\langle (1, m_1), (1, m_2), (1, m_3) \rangle \cong \langle (1, n_1), (1, n_2), (1, n_3) \rangle$ with isomorphism $(1, m_i)\varphi = (1, n_i)$ for $i = 1, 2, 3$.

Further questions

- Q: At what point does the direct product of $\mathbb{N}$ with something become uncountable? Are there countably many subsemigroups of $\mathbb{N} \times S$ for S a finite semigroup?

Theorem D

The following are equivalent for a finite semigroup S:

1. $\mathbb{N} \times S$ has only countably many subsemigroups;
2. $\mathbb{N} \times S$ has only countably many non-isomorphic subsemigroups;
3. S is a union of groups.

Further questions

Theorem E

The following are equivalent for a finite semigroup S:

1. $\mathbb{N} \times S$ has only countably many subdirect products;
2. $\mathbb{N} \times S$ has only countably many non-isomorphic subdirect products;
3. For every $s \in S$, there exists some $t \in S$ such that either $ts = s$ or $st = s$.

- Are there uncountably many subdirect products of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ which project onto pairs?
- Does the above generalise to $\mathbb{N}^k$ projective onto $k-1$ factors?