Generalization of a theorem of Clifford

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Preliminaries

Semigroup S *naturally totally ordered (n.t.o.)*

$$\iff \forall a,b \Rightarrow a|b \vee b|a \;\&\; a|b|a \Rightarrow a=b.$$

Disjoint unions n.t.o semigroups by t.o. sets $\Rightarrow$ *ordinary irreducible n.t.o. semigroups with* ≤ 1 *idempotent* $0(=\infty)$.
Examples: positive but not non-negative cones of t.o. groups, monoids of nonunital principal ideals in valuation rings
a *absorbs* $b \iff ab = a > b$. $I \lhd S$ *absorbent*
$\iff ab = a \;\forall a \in I,\; b \notin I \Rightarrow$
S *disjoint union of n.t.o. ordinary irreducible semigroups indexed by absorbent prime ideals (possibly together with* S *if* $1 \in S$*).*

A result of Clifford

Segment ordinary irreducible n.t.o. nilpotent semigroup with 0.

Theorem 1 (Clifford)

S segment $\Longrightarrow \exists!$ up to isomorphism positive cone $P \subseteq G$, $P_\infty \lhd P, S \cong P/P_\infty, P =< P \setminus P_\infty >,\ G \subseteq (\mathbb{R}^+, >)$ o. group.

Corollary 2 (Short)

Ordinary irreducible n.t.o. semigroups with 0 are precisely multiplicative semigroups of proper principal ideals in valuation rings, roughly speaking, their divisibility theory.

Interaction: divisibility in rings as p. o. semigroups
From now on all structures commutative with either 1 or $0(= \infty)$.

Idea of the proof I

Notation: $X^\star = X \setminus \{1, 0\}$. $F(S^\star)$ free semigroup (without identity) on $S^\star$. Ordinary multiplication on $S^\star$ and nonzero products in $S^\star$ for definition of a congruence relation on $F(X^\star)$ determining P, P_∞.

Observations

No identity in n.t.o. ordinary irreducible semigroups!
Kaplansky: valuation rings factors of valuation domains?
No by Fuchs, Salce, Sheila... in contrast to their divisibility!
Passing process to positive cones in lattice-ordered groups; abstract description of their factors (by what?)
Motivation and hints for solution from ring theory: Bezout rings, factors not necessary Rees factors by ideals but by filters as combination with partial ordering.
Aim: Characterizing factors of non-negative cones in l.o. groups

Bezout monoids

Definition (Boschbach–Ánh–Márki–Vámos)

S *Bezout monoid (B-monoid)* $\iff \exists\, 0 \in S$, natural partial order

1. $\forall a, b \in S : \exists\, \mathrm{GCD}(a, b) = a \wedge b$,
2. $\forall\, a, b, c \in S :\ c(a \wedge b) = ca \wedge cb$,
3. S is *hypernormal*, i.e.,

$$\forall a, b : d = a \wedge b \,\&\, a = da_1 \Rightarrow \exists b_1 : b = db_1 \& a_1 \wedge b_1 = 1,$$

Examples: divisibility theory in arithmetical rings. $3) \Rightarrow$

$$ax = ay \iff \exists u \in a^{\perp} = \{v \in S \mid av = 0\} :\ x \wedge u = y \wedge u$$

Working with Bezout monoids

Appearance of filters instead of ideals because of partial order
Lattice factors instead of Rees factors
Correspondence between B-monoids and arithmetical rings
Quite satisfactory theory for semi-hereditary and semiprime B-monoids using the spectrum of minimal prime filters.

Bezout monoids with one minimal prime filter

Proposition 3

S a B-monoid; M a smallest minimal m-prime filter, $T = S \setminus M$, $Z = \{x \in S \,|\, \exists s \notin M : sx = 0\} \subseteq M; N = M \setminus Z \Rightarrow ZM = 0$; $t < n < z \; \forall \; t \in T$, $n \in N$, $z \in Z$; and T non-negative cone of l.o. group G.

Classical localization $T^{-1}S$ inverting T is not B-monoid.
Divisibility of $T^{-1}S$: the monoid of principal filters in $T^{-1}S$ order-isomorphic to factor Σ of S sending $T \mapsto 1$
Crucial examples: factors of $\mathbb{Z} + x\mathbb{Q}[x]$ by $x^n\mathbb{Q}[x]$ or by $x^n\mathbb{Q}[x] + x^{n-1}\mathbb{Z}[x]$, $n > 1$ and their divisibility theory.

Structural summary

Notation: $X^\bullet = X \dot\cup 0$; $\alpha, \beta, \ldots \in \Sigma = \{a^\sigma = T^{-1}Sa \mid a \in S\}$

$$S_a = S_\alpha = \{b \in S \mid b^\sigma = \alpha = a^\sigma\} \Rightarrow S_1 = T,\ S_0 = Z \Rightarrow$$

$s \in N \Rightarrow S_s \sim G \Rightarrow G$ acts on N. $x^\sigma < y^\sigma \Rightarrow x < y$.

Proposition 4

$xy = y \notin Z \Rightarrow x = 1$, $Y = S \setminus Z \to T^{-1}S$ injective. The filter of $T^{-1}S$ generated by N is exactly $N^\bullet$. $G = < T, T^{-1} >$ acts on N, $a \in N \Rightarrow Ga = S_a$. Divisibility monoid of $T^{-1}S$ is Σ, $S : x^\sigma < y^\sigma \Longrightarrow x < y$. $T^{-1}S$ is $X^\bullet$; $X = G \dot\cup N$, and $Z \neq M \Rightarrow Z$ a factor of G by an appropriate filter.

Factors of nonnegative cones

Theorem 5

S B-monoid; unique minimal m-prime filter $M \neq Z = \{s \in S \,|\, \exists\, t \notin M : ts = 0\} \Rightarrow \exists\, A$ l.o. group; filters $B_\infty \subseteq C_\infty \subseteq P = \{g \in A \mid g \geq 1\}$:

1. $P = < P \setminus C_\infty >$
2. *Rees factors* $P/C_\infty \cong S/Z \Rightarrow S \cong P/C_\infty$ *if* $Z = 0$;
3. $Z \neq 0 :\; S \cong P/B_\infty$ *by* $a \sim b \Longleftrightarrow \exists\, c \in B_\infty : a \wedge c = b \wedge c$.

(1) and (2) determine P, A uniquely up to isomorphism fixing $S \setminus Z$ elementwise by identification of $S \setminus Z$ with $P \setminus C_\infty$.

Theorem 6

S B-monoid; M unique minimal m-prime filter, $T = S \setminus M$, $Z = \{s \,|\, \exists\, t \notin M : ts = 0\}$ factor of the quotient group of $T \Rightarrow S$ factor of nonnegative cone of a l.o. group.

Corollary 7

As above, $|\,\Sigma\,| > 2 \Rightarrow S$ *factor of nonnegative cone of a l.o. group.*

Theorem 8

S *B-monoid;* $M = \{s \in S \,|\, \exists\, t \notin M : ts = 0\} \neq 0$ *unique minimal m-prime filter. If the filter generated by all* $a^{\perp}, (a \in M^{\star})$ *proper, then* S *a factor of nonnegative cone of a l.o. group. More generally, if* $I \vartriangle S$ *m-prime filter in,* $K = \{s \in S \,|\, \exists t \notin I : ts = 0\} \Rightarrow S/K$ *factor of nonnegative cone of a l.o. group.*

Comments and remarks

1. Clifford's result is not a consequence!
2. Segments only subsets in positive cones of t.o. groups.
3. As nonstandard modules, Z only subfactors of l.o. groups over their nonnegative cones.
4. Segments subtracting 0 subsets in t.o. groups but B-monoids with unique minimal prime filters subtracting 0 not subsets in l.o. groups!

Ideas of the proof II

1. Warning: one has to deal with identity!
2. Factors of free monoid generated of $Y = S \setminus Z \subsetneq S^\star$ and $X = G \dot\cup N \nsubseteq S$!
3. Factors not necessarily Rees factors rather induced by lattice structure!
4. Refinements of Clifford's ideas: passing from non-invertible generators in Clifford's proof to ones including invertibles!
5. One has to use both free monoids on X, Y and ordinary multiplication on them,respectively, to define congruences and associated filters.

Basic problem

Hensel's $p-$adic numbers led to find (not necessarily) t.o. groups, monoids as *absolut values* and detailed abstract study of divisibility. After positive cones of t.o. or l.o. groups, respectively, Clifford's result first important progress determining divisibility, i.e., *natural (partial) order* as *good order*.
B-monoids *good choice for absolute value* to classical theory.
Basic problem: construct rings with B-monoids as divisibility.

Solution 1: unique minimal prime filter

Corollary 9

S B-monoid with unique minimal m-prime filter $\Rightarrow \exists$ Bezout ring R with unique minimal prime ideal and S as its divisibility.

Corollary 10

If R is a Bezout ring with one minimal prime ideal I such that the localization R_I is not a field, then the divisibility theory S_R of R is a lattice factor of a positive cone of a lattice-ordered group.

Open question: characterize factors of Bezout domains.
Warning: factors of UFDs are all rings!

Solution 2: semi-hereditary Bezout rings

Definition

B-monoid S semi-hereditary $\iff \forall\, a \in S\ \exists e^2 = e \in S :\ a^{\perp} = Se$.

Theorem 11 (Ánh–Siddoway)

To any semi-hereditary Bezout monoid S there is a semi-hereditary Bezout ring R whose divisibility theory is order-isomorphic to S.

$B = \{e \in S | e^2 = e\}$ Boolean algera, e' complement of $e \in B$. R factor of the localization of contracted monoid algebra at *primitive elements* by relations $e + e' = 1$.

Observations

In contrast to the case of lattice-ordered groups, the construction of R in both cases is not a free construction.

No satisfactory structure theory for semiprime Bezout monoids corresponding to rings of weak dimension ≤ 1

Thank You for Your Attention