A Random Walk Through Random Walks[1]

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Consider a random walk on (the Cayley graph of) a semigroup S with respect to a finite generating set (with multiplicities) X:

- Start at the identity (of S^1 if necessary).
- Right-multiply by elements of X chosen uniformly at random.

Question

How quickly does the random walk "spread out" around S?

Let M be the $S \times S$ transition matrix with entries given by:

$$M_{st} = \frac{|\{x \in X \mid sx = t\}|}{|X|}$$

Then M is a linear (Markov) operator on $\ell_1(S)$.

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Theorem (Kesten 1959)

If S is a group and X is a symmetric generating set, then

- *M is a bounded symmetric operator on $\ell_2(S)$, of norm ≤ 1 (hence spectral radius ≤ 1);*
- *M has spectral radius $1 \iff S$ is* ***amenable****.*

Definition

Recall that S has **bounded indegree** (or **finite geometric type**) if for all x there is a $b \in \mathbb{N}$ such that for all t, at most b elements s satisfy $sx = t$. (Cayley graph vertices have boundedly many edges coming in.)

Proposition

Let S be a finitely generated semigroup and M its right random walk transition matrix. TFAE:

- *S has bounded indegree;*
- *M is an operator on $\ell_2(S)$;*
- *M is a bounded operator on $\ell_2(S)$.*

Remark

Morally, an undefined (or unbounded) operator has spectral radius $\infty > 1$.

Amenable Semigroups

Definition (Day 1957)

A semigroup S is **right amenable** if there is a finitely additive probability measure μ on S such that $\mu(X) = \mu(Xs^{-1})$ for all $X \subseteq S$ and $s \in S$.

$$\left(\text{where } Xs^{-1} \text{ denotes } \{t \in S \mid ts \in X\}\right)$$

Examples

- finite groups (uniform measure)
- **right reversible** (without disjoint left ideals) finite semigroups
- semigroups with 0 $\left(\mu(X) = \begin{cases} 1 & \text{if } 0 \in X \\ 0 & \text{otherwise} \end{cases}\right)$
- commutative semigroups
- the bicyclic monoid

Cogrowth of Groups

Let G be a group generated **as a group** by a finite subset X.
Let $F(X)$ denote the free group on X.

Definition

The **cogrowth function** of G is

$$\kappa \ : \ \mathbb{N} \to \mathbb{N}, \quad n \mapsto |\{w \in F(X) \mid |w| = n, w = 1 \text{ in } G\}|$$

The **cogrowth** of G is

$$\kappa = \limsup_{n \to \infty} \kappa(n)^{1/n}.$$

Theorem (Grigorchuk 1980, Cohen 1982)

$\kappa \leq 2|X| - 1$, *with equality if and only if G is amenable.*

Definition (Gray & K. 2015)

The **local cogrowth function** of S (with respect to X) at s is

$$\lambda_s : \mathbb{N} \to \mathbb{N}, \quad n \to \left| \{ u \in X^+ \mid |u| = n, u = s \text{ in } S \} \right|$$

The **local cogrowth** of S at s is

$$\lambda_s = \limsup_{n \to \infty} \lambda_s(n)^{1/n}.$$

Example

Let $S = B = \langle b, c \mid bc = 1 \rangle$ be the bicyclic monoid, $X = \{b, c\}$.

$$\lambda_1(n) \;=\; \begin{cases} 0 & \text{if } n \text{ is odd} \\ C_{n/2} & \text{if } n \text{ is even.} \end{cases}.$$

From Stirling's approximation, $C_{n/2} \;\approx\; \frac{4^{(n/2)}}{(n/2)^{3/2}\sqrt{\pi}} \;=\; \frac{2^n}{\text{irrelevant rubbish}}$.
So $\lambda_1 = 2$.

Local Cogrowth and Ideals

Proposition

If $s \leq_{\mathcal{J}} t$ *then* $\lambda_s \geq \lambda_t$. *In particular, local cogrowth is a* $\mathcal{J}$*-class invariant.*

Corollary

For all $s \in B$, $\lambda_s = 2$.

Proposition/Exercise

If S *has an element of maximal local cogrowth (i.e. local cogrowth* $|X|$*), then the set of all such elements forms a minimal ideal of* S.

Example

Let $F = \langle a, b \mid ab = ba \rangle$ be the free commutative semigroup of rank 2.

For any w, $\lambda_w(n) = 0$ for all $n > |w|$.

So $\lambda_w = 0$.

Hmmmmmm.......

Definition (Gray & K. 2015)

The **global cogrowth function** of S (with respect to X) is

$$\gamma \ : \ \mathbb{N} \to \mathbb{N}, \quad n \to \left|\{(u,v) \in X^+ \times X^+ \mid |uv| = n, u = v \text{ in } S\}\right|.$$

The **global cogrowth** of S (with respect to X) is

$$\gamma \ = \ \limsup_{n\to\infty} \gamma(n)^{1/n}.$$

Remark

$$\gamma(n) = \sum_{s\in S,\ i+j=n} \lambda_s(i)\lambda_s(j)$$

$$\gamma = \left|\{(u,v) \in X^+ \times X^+ \mid |uv| = n, u = v \text{ in } S\}\right|.$$

$$\gamma = \limsup_{n\to\infty} \gamma(n)^{1/n}.$$

Lemma

One can equivalently replace $\gamma(n)$ with

$$\gamma'(n) = \left|\{(u,v) \in X^+ \times X^+ \mid |u| = |v| = \frac{n}{2}, u = v \ in\ S\}\right|.$$

$\frac{\gamma'(n)}{|X|^n}$ is the prob. that 2 random walks of length $\frac{n}{2}$ end at the same point.

Lemma

For any $0 \leq \kappa < \gamma$ there exists $C > 0$ such that $\gamma(n) \geq \gamma'(n) > C\kappa^n$ for all even n.

Theorem (Gray & K. 2015)

Let S be a semigroup generated by a finite set X. Then

$$\sqrt{|X|} \leq \gamma \leq |X| \quad and$$

$\gamma = \sqrt{|X|} \iff$ *S is a free semigroup freely generated by X or $|X| = 1$.*

Question

When is the global cogrowth **maximal***?*

Question

When is the global cogrowth maximal?

Proposition

Maximal local cogrowth (anywhere) $\implies$ *maximal global cogrowth.*

Theorem (Gray & K. 2015)

If S has subexponential growth then S has maximal global cogrowth.

Corollary

Commutative monoids have maximal global cogrowth (compare local cogrowth).

Example

For the bicyclic monoid, $\gamma = 2$.

Theorem (building on Elder-Rechnitzer-Wong 2012 building on Grigorchuk 1980 / Cohen 1982 building on Kesten 1959)

If S is a group and X is a symmetric generating set then S has maximal global cogrowth if and only if S is amenable.

Question

How sensitive is maximal cogrowth to the choice of generators?

Example

- $G = \mathbb{Z}$ with generators $+1$, -1 has local cogrowth 2.
- $G = \mathbb{Z}$ with generators $+1$, $+1$, -1 has local cogrowth $2\sqrt{2} < 3$.

Theorem (Gray & K. 2017)

Suppose S is a monoid with maximal global cogrowth with respect to some finite choice of generators. Then for every finite $K \subseteq S$, the monoid S has maximal global cogrowth with respect to some choice of generators (with multiplicity) containing K.

Theorem (Gray & K. 2016)

If S has maximal global cogrowth then the associated Markov operator M on $\ell_2(S)$ has spectral radius ≥ 1.

Theorem (Gray & K. 2016)

If S satisfies the right Følner condition then the associated Markov operator M on $\ell_2(S)$ has spectral radius ≥ 1.

As a consequence with have one implication of Kesten's theorem for semigroups:

Corollary

If S is right amenable then the associated Markov operator M on $\ell_2(S)$ has spectral radius ≥ 1.

Theorem (Gray & K. 2015)

Suppose S is right reversible and the max. left cancellative quotient of S has a minimal ideal. If S has maximal global cogrowth then S is right amenable.

Proof.

Uses a theorem of Day (1962) giving "the other" implication of Kesten's theorem for right (!) cancellative monoids. □

Corollary

A finitely generated inverse semigroup (or group!) of maximal global cogrowth is (left and right) amenable. (No symmetry assumption on the generating set!)

Corollary

A finitely generated right reversible semigroup of maximal **local** *cogrowth is right amenable.*

Near Right Cancellativity

Definition

$E \subseteq S$ is **right thick** if for all finite $F \subseteq S$ there exists $t \in S$ with $tF \subseteq E$.

Definition (Gray & K. 2015)

S is **near right cancellative** if for all $s \in S$ there exists a right thick $E \subseteq S$ such that $\forall x, y \in E$, $xs = ys \implies x = y$.

Examples

- right cancellative semigroups ($E = S$)
- semigroups with 0 ($E = \{0\}$)
- inverse semigroups
- right reversible semigroups where every ideal contains an idempotent

Remark

These semigroups **behave dynamically** *like right cancellative semigroups.*

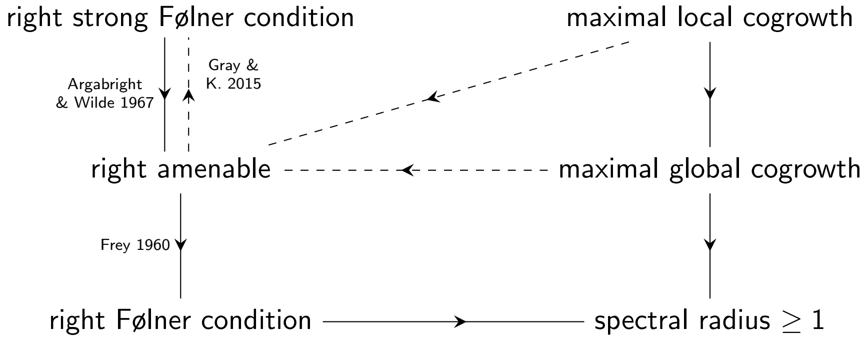
Definition

S is **near right cancellative** if for all $s \in S$ there exists a right thick $E \subseteq S$ such that $\forall x, y \in E$, $xs = ys \implies x = y$.

Theorem (Gray & K. 2017)

If S is a right reversible, near right cancellative monoid of maximal global cogrowth then S is right amenable.

The Big Picture



$\longrightarrow$ implications hold for all semigroups.

$\dashrightarrow$ implications hold for right reversible near right cancellative semigroups.

More detail....

- R.D.Gray and M.Kambites, *Amenability and geometry of semigroups*, Trans. AMS (to appear), preprint at `arXiv:1505.06139 [math.GR]`
- R.D.Gray and M.Kambites, *On cogrowth, amenability and the spectral radius of a random walk on a semigroup*, preprint at `arXiv:1706.01313 [math.GR]`

Also relevant....

- **P.Gerl (1973).** Can be interpreted as connecting left amenability and maximum local cogrowth where S is left cancellative with a left identity.
- Lots of work on random walks on semigroups and monoids from different perspectives.
- Other ways to define amenability for inverse semigroups.