Cohomology and extensions of inverse semigroups

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Debts of honour

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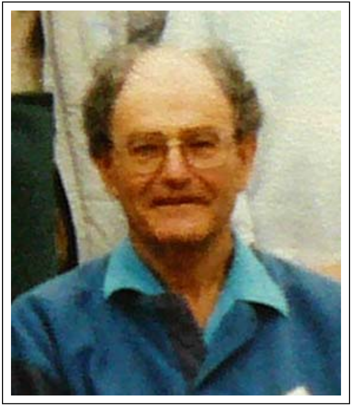


Karl Gruenberg

Forthcoming attractions

- ~~Lunch~~
- Extensions of groups and factor sets
- Extensions of inverse semigroups
- Cohomology of inverse semigroups
- Classification

The extension problem

A group G is an *extension* of K by Q if G contains a normal subgroup (isomorphic to) K with G/K isomorphic to Q.

Sometimes called an extension of Q by K but this is just wrong.

Represent by a short exact sequence

$$1 \to K \to G \to Q \to 1\,.$$

Extension problem: given K and Q, classify all the extensions.

(Pre)–history

- Schreier 1926, Baer 1934, Turing 1938
- An extension has a *coupling* χ:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & K & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \chi} & & \\
1 & \longrightarrow & \mathrm{Inn}\, K & \longrightarrow & \mathrm{Aut}\, K & \longrightarrow & \mathrm{Out}\, K & \longrightarrow & 1
\end{array}$$

  Refined extension problem: given K and Q, classify all the extensions with a given coupling.

- Construct multiplication table for G from K and a transversal.
- Simplified when K is abelian, and so is a Q–module with $\chi : Q \to \mathrm{Aut}\, K$

Abelian kernel

Extension with abelian kernel A:

$$1 \to A \to G \to Q \to 1$$

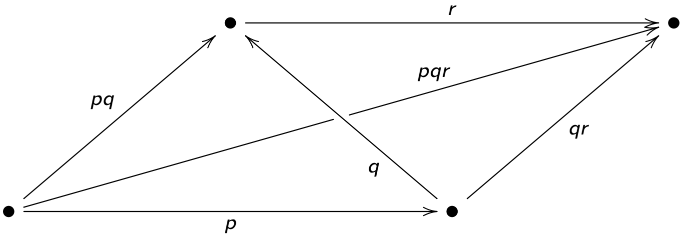
with $\{\overline{q} : q \in Q\}$ transversal to A in G. Each $g \in G$ is then uniquely $g = \overline{p}a$ $(p \in Q, a \in A)$.

$$\begin{aligned} \overline{p}a \cdot \overline{q}b &= \overline{p} \cdot \overline{q} \cdot [\overline{q}^{-1} a \overline{q}] \cdot b \\ &= \overline{pq} \cdot ((p,q)f \cdot a^q \cdot b) \end{aligned}$$

where $f : Q \times Q \to A$, defined by $\overline{p} \cdot \overline{q} = \overline{pq} \cdot (p,q)f$, is a *factor set*. *Associativity* in G then implies

$$(pq,r)f \cdot (p,q)f^r = (p,qr)f \cdot (q,r)f \, .$$

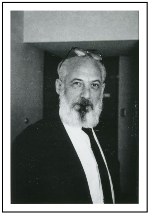
Geometry of factor sets

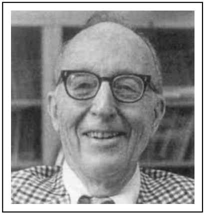


Front face represents product of p and q, written $|p|q|$ and so on, and boundary of tetrahedron (based at right-most vertex) is

$$\text{front - base + left -right} = |p|q| \triangleleft r - |p|qr| + |pq|r| - |q|r| \,.$$

Homological (pre)–history





- Eilenberg & MacLane 1942 *...the theory of group extensions forms a natural and powerful tool in the study of homologies in ...topological spaces*
- identified factor sets as 2–*cocycles*, representing elements of $H^2(Q, A)$
- extensions are classified by a second cohomology group, via factor sets

You say you want a resolution ...

Compute cohomology $H^n(Q, A)$ using the bar resolution. This is the *nerve* of G considered as a category with one object.

$$F_n = \text{free } Q\text{–module on basis } \{|q_1|\cdots|q_n| : q_i \in Q\}$$

and $\partial : F_3 \to F_2$ by

$$|p|q|r| = |p|q|^r - |p|qr| + |pq|r| - |q|r| \, .$$

Get induced map $\partial^* : \mathrm{Hom}_Q(F_2, A) \to \mathrm{Hom}_Q(F_3, A)$ and $f : Q \times Q \to A$ is in $\ker \partial^*$ iff

$$(p, q)f^r - (p, qr)f + (pq, r)f - (q, r)f = 0$$

for all $p, q, r \in Q$. Now recall factor set condition

$$(pq, r)f \cdot (p, q)f^r = (p, qr)f \cdot (q, r)f \, .$$

Inverse semigroups

In an *inverse* semigroup Q, each $q \in Q$ has a *unique* $q^{-1} \in Q$ such that

$$qq^{-1}q = q \text{ and } q^{-1}qq^{-1} = q^{-1}\,.$$

Equivalently, there exists $p \in S$ with $qpq = q$ and the idempotents in Q commute: qq^{-1} is an idempotent.

Theorem

A semigroup Q, in which each $q \in Q$ has a unique q' such that $qq'q = q$, is a group.

Partial order on Q determined by semilattice of idempotents $E(Q)$:

$$p \leqslant q \iff \exists e \in E(Q) \ : \ p = eq\,.$$

Extensions of inverse semigroups I

An extension is now

$$K \to S \xrightarrow{\theta} Q$$

with θ *idempotent-separating*: so $\theta : E(S) \to E(Q)$ is bijective. Then $K = \{s \in S : s\theta \in E(Q)\}$ a semilattice of groups.

Schreier's approach followed by Coudron (1968) and d'Alarcao (1969), then by Lausch, *Cohomology of inverse semigroups*, J. Algebra (1975):

> *Whereas Eilenberg and MacLane could phrase the theory of group extensions in terms of cohomology ... the extension problem for inverse semigroups was left in the wilderness ...*

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Extensions of inverse semigroups II

Lausch's approach recast by Loganathan, *Cohomology of inverse semigroups*, J. Algebra (1981) in terms of cohomology of categories, but 1982 applications to extensions still reliant on factor sets.

In place of a wilderness we now have growing interest in cohomology of categories:

- Baues & Wirsching 1985
- Hoff 1991
- Webb & Xu 2007, 2011
- Linckelmann 2013

Extensions and cohomology

We'll now discuss extensions of inverse semigroups, but aim for an approach to extensions via their overall structure, not multiplication tables.

We follow Gruenberg 1967 for groups: as Webb (2011) writes:

> *It will be apparent . . . that much of what I have done is to present [Gruenberg's] work in the context of [inverse semigroups]. The influence of Gruenberg's development of the theory is pervasive . . .*

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Ingredients

- inverse *monoid* Q (for good technical reasons)
- an *Q–module* $\mathscr{A}$ is a semilattice of abelian groups $\{A_e : e \in E(Q)\}$ with homomorphisms $\alpha^e_f : A_e \to A_f$ whenever $e \geqslant f$, and for each $q \in Q$, an isomorphism $\gamma_q : A_{qq^{-1}} \to A_{q^{-1}q}$
- module $\mathbb{Z}Q$ at $e \in E(Q)$ has free abelian group on Green's $\mathcal{L}$–class of e.
- Loganathan: $\mathbb{Z}Q$ is a projective Q–module if and only if Q is a monoid.
- Q–module $\mathbb{Z}$ has group $\mathbb{Z}$ at every $e \in E(Q)$.

Making extensions I

For a Q–module $\mathscr{A}$,

$$S = Q \ltimes \mathscr{A} = \{(p, a) : a \in A_{p^{-1}p}\}$$

with

$$\begin{aligned}(p, a)(q, b) &= (pq, [a\alpha^{p^{-1}p}_{p^{-1}pqq^{-1}}\gamma_{p^{-1}pq}] + b\alpha^{q^{-1}q}_{q^{-1}p^{-1}pq}) \\ &= (pq, a \triangleleft q + b)\end{aligned}$$

is an inverse semigroup, and is the *split extension* of $\mathscr{A}$ by Q.

Making extensions II

Take any free inverse monoid $\mathbb{F}$ mapping onto Q and factorize $\theta : \mathbb{F} \to Q$ as

$$\mathbb{F} \to \mathbb{T} \xrightarrow{\psi} Q$$

with ψ idempotent-separating and $\mathbb{T}$ maximal:

$$\mathbb{T} = \mathbb{F}/\tau \text{ where } u\tau v \iff \exists w : u \geqslant w \leqslant v \text{ and } u\theta = w\theta = v\theta \,.$$

We get an extension

$$\mathbb{U} \to \mathbb{T} \xrightarrow{\psi} Q$$

where $\mathbb{U}$ is generally a semilattice of non-abelian groups, acted on by $\mathbb{T}$ by conjugation. Abelianising each U_e gives a Q–module $\mathbb{U}^{ab}$.

Any Q–module $\mathscr{A}$ is a $\mathbb{T}$–module via ψ, so we can form $\mathbb{S} = \mathbb{T} \ltimes \mathscr{A}$.

Making extensions III

For any $\mathbb{T}$–map $\phi : \mathbb{U} \to \mathscr{A}$, we get a congruence on $\mathbb{S}$ induced by left multiplication by the inverse subsemigroup

$$K_\phi = \{(u, u\phi) : u \in \mathbb{U}\} \subseteq \mathbb{S}\,.$$

Theorem

There is an extension

$$\mathscr{E}_\phi : \quad \mathscr{A} \to \mathbb{S}/K_\phi \to Q$$

and (up to equivalence) every extension of $\mathscr{A}$ by Q arises in this way.

Making extensions IV

Given an extension $\mathscr{E} : \quad \mathscr{A} \to S \to Q$:

Lift $S \to Q$ to $\mathbb{F} \to S$ by freeness: this factors through $\varphi : \mathbb{T} \to S$ by maximality of $\mathbb{T}$:

$$\begin{array}{ccccc} \mathbb{U} & \longrightarrow & \mathbb{T} & \xrightarrow{\psi} & Q \\ \downarrow & & \downarrow \varphi & & \| \\ \mathscr{A} & \longrightarrow & S & \longrightarrow & Q \end{array}$$

giving a $\mathbb{T}$–map $\varphi : \mathbb{U} \to \mathscr{A}$, and $\mathscr{E}$ is equivalent to $\mathscr{E}_\varphi$.
Lift $q \in Q$ to $\overline{q} \in \mathbb{T}$:

Theorem

$(p, q) \mapsto ((\overline{pq})\varphi)^{-1}\, \overline{p}\varphi\, \overline{q}\varphi$ is a factor set for $\mathscr{E}$.

Where's ~~Wally~~ the cohomology?

We compute cohomology using projective Q–modules: have exact sequence

$$0 \to \mathbb{U}^{ab} \to \mathscr{D} \to \mathbb{Z}Q \to \mathbb{Z} \to 0$$

with $\mathscr{D}$ and $\mathbb{Z}Q$ projective, and so get exact

$$\mathrm{Hom}_Q(\mathscr{D}, \mathscr{A}) \to Hom_Q(\mathbb{U}^{ab}, \mathscr{A}) \to H^2(Q, \mathscr{A}) \, .$$

Image of $\mathrm{Hom}_Q(\mathscr{D}, \mathscr{A})$ exactly corresponds to equivalence of extensions derived from $\mathbb{T}$–maps $\mathbb{U} \to \mathscr{A}$ and so

Theorem

The idempotent separating extensions of a Q–module $\mathscr{A}$ by Q are classified by the cohomology group $H^2(Q, \mathscr{A})$.

What is $\mathscr{D}$?

Equivalence of extensions $\mathscr{E}_\alpha \leftrightarrow \mathscr{E}_\beta$ is given by a homomorphism $\mu : S_1 \to S_2$:

$$\begin{array}{ccccc} \mathscr{A} & \longrightarrow & \mathbb{S}/K_\alpha & \longrightarrow & Q \\ \| & & \downarrow{\scriptstyle \mu} & & \| \\ \mathscr{A} & \longrightarrow & \mathbb{S}/K_\beta & \longrightarrow & Q \end{array}$$

and $\alpha - \beta : \mathbb{U} \to \mathscr{A}$ is then a restriction to $\mathbb{U}$ of a derivation : $\partial : \mathbb{T} \to \mathscr{A}$: for $[(u, u^{-1}u)]_\alpha \in \mathbb{S}/K_\alpha$,

$$[u, u^{-1}u]_\beta([u, u^{-1}u]_\alpha)^{-1}\mu = [u^{-1}u, a]_\beta$$

for some $a \in A_{u^{-1}u}$, and $\partial : u \mapsto a$. $\mathscr{D}$ is the universal Q–module that turns derivations into Q–maps $\mathscr{D} \to \mathscr{A}$.