Version of May 16, 2016

THE SEMIGROUP βS

If S is a discrete space, its Stone-Čech compactification βS can be described as the space of ultrafilters on S with the topology for which the sets of the form $\overline{A} = \{p \in \beta S : A \in p\}$, where $A \subseteq S$, is chosen as a base for the open sets. (Note that we embed S in βS by identifying $s \in S$ with the principal ultrafilter $\{A \subseteq S : s \in A\}$.)

βS is then an extremally disconnected compact space and $\overline{A} = cl_{\beta S}(A)$ for each $A \subseteq S$.

If S is a semigroup, the semigroup operation on S has a natural extension to βS.

Given $s \in S$, the map $t \mapsto st$ from S to βS has a continuous extension to βS, which we denote by λ_s. For $s \in S$ and $q \in \beta S$, we put $sq = \lambda_s(q)$. Then, for every $q \in \beta S$, the map $s \mapsto sq$ from S to βS has a continuous extension to βS, which we denote by ρ_q. We put $pq = \rho_q(p)$. So $pq = \lim_{s \to p} \lim_{t \to q} st$.

It is easy to see that this operation on βS is associative, so that βS is a semigroup. It is a right topological semigroup, because ρ_q is continuous for every $q \in \beta S$. In addition, λ_s is continuous for every $s \in S$. These two facts are summed up by saying that βS is a semigroup compactification of S. It is the maximal semigroup compactification of S, in the sense that every other semigroup compactification of S is the image of βS under a continuous homomorphism.

We shall use S^* to denote the remainder space $\beta S \setminus S$.

If S and T are semigroups, every homorphism from S to T extends to a continuous homomorphism from βS to βT.

If T is a subset of a semigroup, $E(T)$ will denote the set of idempotents in T.

Every compact right topological semigroup T has important algebraic properties. I shall need to use the following:

(i) T contains an idempotent; i.e. an element p for which $p^2 = p$.

(ii) A non-empty subset V of T is said to be a *left ideal* if $TV \subseteq V$ and a *right ideal* if $VT \subseteq V$. It is an *ideal* if it is both a left and a right ideal. T contains a smallest ideal $K(T)$, which is the union of all its minimal left ideals and the union of all its minimal right ideals. If L is a minimal left ideal and R a minimal right ideal of T, then $L \cap R = RL$ is a group.

(iii) $K(T)$ always contains an idempotent. An idempotent in $K(T)$ is called *minimal.* An idempotent in T is minimal in this sense if and only if it also minimal for the partial order defined on idempotents by putting $p \leq q$ if and only if $pq = qp = p$. If p is any idempotent in T, there is an idempotent $q \in K(T)$ satisfying $q \leq p$. We also define quasi-orders $\leq_L$ and $\leq_R$ on the idempotents of T by putting $p \leq_L q$ if $pq = p$ and $p \leq_R q$ if $qp = p$.

(iv) If S is a discrete semigroup, a subset of S is said to be *central* if it is a member of a minimal idempotent in βS. Central sets have very rich combinatorial properties.

APPLICATIONS TO RAMSEY THEORY

Ramsey Theory is the study of properties of finite partitions of a given set. We shall often refer to a finite partition of a set S as a *finite colouring* of S, and call a subset of S *monochrome* if it is contained in a cell of the partition.

Observe that, given any finite colouring of S and any ultrafilter $p \in \beta S$, p will have a member that is monochrome.

HINDMAN'S THEOREM

Notation

Given a sequence (x_n) in a semigroup, $FP\langle x_n\rangle$ denotes the set of all products of the form $x_{n_1}x_{n_2}\cdots x_{n_k}$ with $n_1 < n_2 < \cdots < n_k$. (If S is denoted additively, we might denote this set by $FS\langle x_n\rangle$.)

If S is a semigroup, p is an idempotent in βS and $A \in P$, then $A^\star = \{s \in A : s^{-1}A \in p\}$, where $s^{-1}A = \{t \in S : st \in A\}$. It is easy to show that $A^\star \in p$ and that $t^{-1}A^\star \in p$ for every $t \in A^\star$.

Hindman′sTheorem

Let S be a semigroup. Given any finite colouring of S, there is a sequence $(x_n)_{n-1}^{\infty}$ in S such that $FP\langle x_n\rangle$ is monochrome.

Ultrafilterproof (Galvin Glazer)

I shall show that, if p is an idempotent in βS and $A \in p$, then $FP\langle x_n\rangle \subseteq A$ for some sequence (x_n) in S.

Choose any $x_1 \in A^\star$. Then assume that $x_1, x_2, \cdots, x_n$ have been chosen so that $FP\langle x_i\rangle_{i=1}^{n} \subseteq A^\star$. Choose $x_{n+1} \in A^\star \cap \bigcap_{y\in FP\langle x_i\rangle} y^{-1}A^\star$. This is possible, because this set is a finite intersection of elements of p and is therefore non-empty. Then $FP\langle x_i\rangle_{i=1}^{n+1} \subseteq A^\star$. □

Note that, if $p \in \beta S \setminus S$, $\langle x_n\rangle$ can be chosen as a sequence of distinct points.

THEOREM

Given a finite colouring of $\mathbb{N}$, there exist infinite sequences (x_n) and (y_n) in $\mathbb{N}$ such that $FS\langle x_n\rangle \cup FP\langle y_n\rangle$ is monochrome.

Proof

There is an idempotent p in $K(\mathbb{N},\cdot)$ which is in the closure of the idempotents in $K(\beta\mathbb{N},+)$.

This follows from the fact that the closure of the minimal idempotents in $(\beta\mathbb{N},+)$ is a left ideal in $(\beta\mathbb{N},\cdot)$.

So every member of p is also a member of an idempotent in $(\beta\mathbb{N},+)$. $\square$

VAN DER WAERDEN'S THEOREM

Theorem

Let $(S,+)$ be a commutative semigroup. Given any finite colouring of S, there is an arbitrarily long AP which is monochrome.

Proof

We shall show that, if $p \in K(\beta S)$ and $A \in p$ then A contains arbitrarily long AP's.

Let $\ell \in \mathbb{N}$ and put $T = (\beta S)^\ell$. Put $\tilde{p} = (p,p,p,\cdots,p) \in T$. We define subsets E and I of S^ℓ as follows:

$$\begin{array}{rc} I = & \{(a, a+d, a+2d, \cdots, a+(\ell-1)d) : a, d \in S\} \\ E = & \{(a,a,a,\cdots,a) : a \in S\} \cup I \end{array}.$$

Then E is a subsemigroup of T and I is an ideal in E.

Furthermore, $\overline{E}$ is a subsemigroup of T and $\overline{I}$ is an ideal in $\overline{E}$. Now $\tilde{p} \in \overline{E}$ and it follows easily that $\tilde{p} \in K(\overline{E})$. So $\tilde{p} \in \overline{I}$. Since $\overline{A}^\ell$ is a neighbourhood of $\tilde{p}$ in T, $\overline{A}^\ell \cap I = A^\ell \cap I \neq \emptyset$. So there exist $a, d \in S$ such that $(a, a+d, a+2d, \cdots, a+(\ell-1)d) \in A^\ell$. $\square$

COROLLARY

Given a finite colouring of $\mathbb{N}$, there is an arbitrarily long AP A and an arbitrarily long GP G such that $A \cup G$ is monochrome.

Proof

We can choose $p \in K(\beta\mathbb{N},\cdot) \cap \overline{K(\beta\mathbb{N},+)}$. Then every member of p contains arbitrarily long AP's and arbitrarily long GP's. $\square$

THE HALES JEWETT THEOREM

Theorem

Let A denote a finite alphabet and let v denote any element which is not in A. Let S denote the semigroup of words over A, and let $S(v)$ denote the semigroup of words over $A \cup \{v\}$ which contain v. Let $W = S \cup S(v)$. For each $a \in A$ and $w \in W$, let $w(a) \in S$ be defined as the word obtained from w by replacing all occurrences of v by a. Then given any finite colouring of S, there exists $w \in S(v)$ such that $\{w(a) : a \in A\}$ is monochrome.

Proof (A. Blass)

Define $h_a : W \to S$ by $h_a(w) = w(a)$. Observe that h_a is a homomorphism, and hence that h_a extends to a continuous homomorphsim from βW onto βS. Choose a minimal idempotent $p \in \beta S$ and a minimal idempotent $q \in \beta W$ satisfying $q \leq p$. For each $a \in A$, $h_a(q) \leq h_a(p) = p$. So $h_a(q) = p$. Hence, given any $P \in p$, there exists $Q \in q$ such that $h_a(Q) \subseteq P$. If $w \in Q$, then $w(a) \in P$ for every $a \in A$. $\square$

EXTENSION OF VAN DE WAERDEN'S THEOREM (I.Leader, N.Hindman)

Note that if $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ \vdots & \vdots \\ 1 & \ell-1 \end{pmatrix}$, then an AP can be described as the set of entries of a column vector of the form $A\begin{pmatrix} a \\ d \end{pmatrix}$.

Let S be a commutative semigroup. There is a set of matrices $\mathcal{A}$ over ω with the following property: If $A \in \mathcal{A}$ and C is a central subset of S, then C contains all the entries of AX for some column vector X over S for which AX is defined. $\mathcal{A}$ contains all matrices over ω, with no row identically zero, in which the first non-zero entries in two different rows are equal if they occur in the same column. We also require that cS is a central subset of S whenever c is the first non-zero entry of a row of A.

In particular, $\mathcal{A}$ contains all finite matrices over ω, with no row identically zero, in which the first non-zero entry of each row is 1.

So if $A \in \mathcal{A}$, in every finite colouring of S, there is a column matrix X with entries in S such that AX is defined and all the entries of AX are monochrome. A matrix A with these properties is called *image partition regular*.

A finite matrix A over $\mathbb{Q}$ is image partition regular if and only if every central subset of $\mathbb{N}$ contains all the entries of AX for some column matrix X over $\mathbb{Q}$ for which AX is defined. In particular, finite image partition martrices over $\mathbb{Q}$ can be diagonalised, in the sense that, whenever A and B are two matrices of this kind, then $\begin{pmatrix} A & O \\ O & B \end{pmatrix}$ is also image partition regular.

<u>ANOTHER EXTENSION</u> (V. Bergelson)

Every central subset C of $(\mathbb{N}, \cdot)$ contains an arbitrarily long geoarithmetic progression. I.e., given $\ell \in \mathbb{N}$, there exist $a, b, d \in \mathbb{N}$ such that $b(a + id)^j \in C$ for every $i, j \in \{0, 1, 2, \cdots, \ell\}$.

<u>FURTHER EXTENSIONS</u> (M. Beiglböck, V. Bergelson, N. Hindman, DS)

If S is a commutative semigroup and $\mathcal{F}$ a partition regular family of finite subsets of S, then for any finite partition of S and any $k \in \mathbb{N}$, there exists $b, r \in S$ and $F \in \mathcal{F}$ such that $rF \cup \{b(rx)^j : x \in F, j \in \{0, 1, 2, \ldots, k\}\}$ is contained in a cell of the partition.

Let $\mathcal{F}$ and $\mathcal{G}$ be families of subsets of $\mathbb{N}$ such that every multiplicatively central subset of $\mathbb{N}$ contains a member of $\mathcal{F}$ and every additively central subset of $\mathbb{N}$ contains a member of $\mathcal{G}$. If either $\mathcal{F}$ or $\mathcal{G}$ is a family of finite sets, then, given any finite colouring of $\mathbb{N}$, there exists $B \in \mathcal{F}$ and $C \in \mathcal{G}$ such that $B \cup C \cup B \cdot C$ is monochrome.

MILLIKEN TAYLOR SYSTEMS

The theory of the partition regularity of finite systems of linear equations is well understood. Given a finite matrix over a field, the question of whether it is image partition regular has a computable answer. Infinite systems present far greater difficulty. Milliken Taylor systems are among the small number of infinite systems known to be image partition regular. Suppose that $\langle a_1, a_2, \ldots, a_n \rangle$ ia a finite sequence of nonzero integers, with successive terms distinct. The Milliken Taylor matrix $M = MT\langle a_1, a_2, \ldots, a_n \rangle$ is an $\omega \times \omega$ matrix which contains all possible rows satisfying the following conditions:

(i) There are only a finite number of non-zero entries in each row;

(ii) No row is identically zero;

(iii) The non-zero entries in each row lie in $\{a_1, a_2, \ldots, a_n\}$, with each a_i occurring and each occurrence of a_i preceding each occurrence of a_{i+1}.

The Milliken Taylor Theorem states that, in any finite colouring of $\mathbb{Z}$, there is an $\omega \times 1$ matrix $\vec{x}$ with integer entries such that all the entries of $M\vec{x}$ are monochrome. In fact, if p is any idempotent in $\beta\mathbb{Z}$ and P is any member of p, the entries of $\vec{x}$ can be chosen to lie in P.

Note that Hindman's Theorem is a special case of this theorem, because Hindman's Theorem follows from the partition regularity of $M\langle 1 \rangle$, the finite sums matrix.

Two different MT matrices are incompatible. If $A = MT\langle\vec{a}\rangle$ and $B = MT\langle\vec{b}\rangle$ are MT matrices, where $\vec{a}$ and $\vec{b}$ are not rational multiples of each other, there is a two colouring of $\mathbb{Z}$ for which there do not exist $\omega \times 1$ matrices $\vec{x}$ and $\vec{y}$ over $\mathbb{Z}$ for which all the entries of $A\vec{x}$ and $B\vec{y}$ have the same colour. So infinite image partition regular matrices over $\mathbb{Q}$ cannot be diagonalised.

However, translating these matrices completely changes the situation. A recent result, due to N. Hindman, I. Leader and DS, shows that if $M = MT\langle a_1, a_2, \ldots, a_n\rangle$, where $a_n = 1$, and if $H = MT\langle 1\rangle$, then the matrix $A = \begin{pmatrix} \overline{1} & M \\ \overline{0} & H \end{pmatrix}$ is partition regular. (Here $\overline{a}$ denotes the constant $\omega \times 1$ matrix whose entries are all equal to a.) In fact, given any central subset C of $\mathbb{N}$, there exists a column vector X with entries in $\mathbb{Z}$ for which all the entries of AX are in C.

More generally, if Millken Taylor $A = MT\langle a_1, a_2, \ldots a_n\rangle$ and $B = MT\langle b_1, b_2, \ldots, b_k\rangle$, then $\begin{pmatrix} \overline{1} & A \\ \overline{0} & B \end{pmatrix}$ is image partition regular provided that $a_n = b_k$.

ADDITIVE AND MULTIPLICATIVE IDEMPOTENTS IN $\beta\mathbb{N}$

THEOREM (DS)

The closure of the smallest ideal of $(\beta\mathbb{N}, \cdot)$ does not meet the smallest ideal of $(\beta\mathbb{N}, +)$. In fact, it does not meet $\mathbb{N}^* + \mathbb{N}^*$.

THEOREM (DS) The closure of the set of multiplicative idempotents in $\beta\mathbb{N}$ does not meet the set of additive idempotents.

Lemma 1

Let A and B be σ-compact subsets of a compact F-space. Then $\overline{A} \cap \overline{B} \neq \emptyset$ implies that $\overline{A} \cap B \neq \emptyset$ or $A \cap \overline{B} \neq \emptyset$.

Lemma 2

Let $\mu\mathbb{R}$ denote the uniform compactification of $\mathbb{R}$. This is a compact right topological semigroup in which $\mathbb{R}$ is densely embedded, with the defining property that a bounded continuous real function has a continuous extension to $\mu\mathbb{R}$ if and and only if it is uniformly continuous.

The log function from $\mathbb{N}$ to $\mathbb{R}$ has a continuous extension to a function L from $\beta\mathbb{N}$ to $\mu\mathbb{R}$. L has the following properties:

(i) $L(x+y) = L(y)$ for every $x \in \beta\mathbb{N}$ and every $y \in \mathbb{N}^*$.

(ii) $L(xy) = L(x) + L(y)$ for every $x, y \in \beta\mathbb{N}$.

Remark

For $x \in \beta\mathbb{N}$ and $n \in \mathbb{N}$, nx will denote $\lim_{s\to x} ns$. Note that this is not the same as $x + x + \ldots + x$, with n terms in the sum.

Proof of Theorem

Let $\mathbb{H} = \bigcap_{n\in\mathbb{N}} cl_{\beta\mathbb{N}}(2^n\mathbb{N})$.

Let $\mathbb{T}$ denote the unit circle.

Observe that $\mathbb{H}$ contains all the idempotents in $(\beta\mathbb{N}, +)$ and that every idempotent in $(\beta\mathbb{N}, \cdot)$ is either in $\mathbb{H}$ or in $cl_{\beta\mathbb{N}}(2\mathbb{N}-1)$.

Let $C = cl_{\beta\mathbb{N}}(E(\beta\mathbb{N}, \cdot)) \cap \mathbb{H}$. Assume that there exists $p \in E(\beta\mathbb{N}, +) \cap C$.

Let $D = \{x \in \mu\mathbb{R} : \phi(x) = 0$ for every continuous homomorphism $\phi : \mu\mathbb{R} \to \mathbb{T}\}$. Then $L(C) \subseteq D$ and so $L(p) \in D$. Observe that, for every distinct $s \neq 0$ in $\mathbb{R}$, $(s+D) \cap D = \emptyset$. It follows that, for any $n > 1$ in $\mathbb{N}$, $L(p) \notin L(n) + D$.

We have $p \in cl_{\beta\mathbb{N}}((\mathbb{N}\setminus\{1\}) + p)$. We also have $p \in cl_{\beta\mathbb{N}}(\bigcup\{nC : n \in \mathbb{N}, n > 1\})$, because $E(\beta\mathbb{N}, \cdot) \cap \mathbb{H} \subseteq cl_{\beta\mathbb{N}}(\bigcup\{nC : n \in \mathbb{N}, n > 1\})$.

It follows from Lemma 2 that $x + p \in nC$ for some $x \in \beta\mathbb{N}$ and some $n > 1$ in $\mathbb{N}$, or else $n + p \in cl_{\beta\mathbb{N}}(\bigcup\{nC : n \in \mathbb{N}, n > 1\})$.

The first possibility is ruled out because it implies that $L(p) \in L(n)+D$. The second is ruled by the observation that $n+p \notin \mathbb{H}$, while $nC \subseteq \mathbb{H}$ for every $n \in \mathbb{N}$. □

COROLLARY

There is no idempotent $p \in (\beta\mathbb{N}, +)$ such that every member of p contains all the finite products of an infinite sequence in $\mathbb{N}$.

QUESTION

Is there an idempotent $p \in (\beta\mathbb{N}, +)$ such that every member of p contains three integers of the form x, y, xy?

SOME PROPERTIES OF IDEMPOTENTS IN $\beta\mathbb{N}$

(1) (N. Hindman, DS) There are $2^{\mathfrak{c}}$ idempotents in $\overline{K(\beta\mathbb{N})} \setminus K(\beta\mathbb{N})$.

(2) (N. Hindman, DS, Y. Zelenyuk) $\beta\mathbb{N}$ contains decreasing $\leq_L$ chains of idempotents indexed by $\mathfrak{c}$. If α is a countable ordinal, $\beta\mathbb{N}$ contains decreasing chains of idempotents indexed by α.

(3) (N. Hindman, DS) $\beta\mathbb{N}$ contains increasing chains $\leq_R$ chains of idempotents indexed by ω_1.

(4) (Y. Zelenyuk) $K(\beta\mathbb{N})$ contains rectangular semigroups of cardinality $2^{\mathfrak{c}}$. (A rectangular semigroup is one in which every element is idempotent and the identity $xyz = xz$ is satisfied.)

(5) Martin's Axiom implies that $\beta\mathbb{N}$ contains idempotents which have a basis consisting of finite sum sets; but this cannot be proved in ZFC. The existence of an idempotent of this kind implies the existence of an infinite extremally disconnected Boolean topological group.