Semigroup algebra of a restriction semigroup with an inverse skeleton

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York, May 11, 2016

Joint work with G. M. S. Gomes and F. Soares
Starting point

$R$ ring, $G$ group, $R(G)$ group ring

Well studied (Connell, Passman, ...):
- $R(G)$ prime, semiprime
- $R(G)$ primitive, semiprimitive

Theorem (Domanov, 76)

$F$ field, $S$ inverse semigroup

If $F(G)$ is semiprimitive for every nonzero maximal subgroup $G$ of $S$, then $F(S)$ is semiprimitive.

Theorem (Domanov, 76)

$F$ field, $S$ inverse semigroup

If $S$ is 0-bisimple and $F(G)$ is primitive for every nonzero maximal subgroup $G$ of $S$, then $F(S)$ is primitive.

Converse false (Teply, Turman and Quesada, 80).
A ring, not necessarily with identity

The ring $A$ is *prime* if for all (left, right, two-sided) ideals $I$ and $J$ of $A$ such that $IJ = 0$, then either $I = 0$ or $J = 0$.

The ring $A$ is *semiprime* if for any (left, right, two-sided) ideal $I$ of $A$ such that $I^2 = 0$, then $I = 0$. 
Primitivity and Semiprimitivity

$M$ right $A$-module

The set $(0: M) = \{ a \in A: Ma = 0 \}$ is called the (right) annihilator of $M$ and is an ideal of $A$.

$M$ is faithful if $(0: M) = 0$.

$M$ is simple if $M \neq 0$ and $M$ has no proper submodules.

$M$ is semisimple if it is the direct sum of simple submodules.

The ring $A$ is right primitive if it admits a simple faithful right module.

The ring $A$ is semiprimitive if it admits a semisimple faithful right module.
Remarks

- Semiprimitivity is a left-right symmetric concept.
- Primitivity is not left-right symmetric.
- Every primitive ring is prime and semiprimitive.
- Both prime and semiprimitive rings are semiprime.
Jacobson radical

An element $a \in A$ is left quasiregular if there exists $r \in A$ such that $r + a + ra = 0$.

A (left, right or two-sided) ideal $I$ of $A$ is said to be left quasiregular if every element of $I$ is left quasiregular.

Right quasiregular elements and right quasiregular ideals are defined analogously.

The Jacobson radical $J(A)$ of $A$ can be characterized as the (left, right) quasiregular (left, right) ideal of $A$ which contains every (left, right) quasiregular ideal.

Recall: $A$ is semiprimitive if and only if $J(A) = 0$. 
Contracted semigroup ring

$S$ semigroup with zero, $R$ ring with identity

The set of finite formal sums

$$\sum_{x \in S} \alpha_x x$$

with coefficients in $R$, equipped with the obvious definition of addition and multiplication, is the \textit{semigroup ring of $S$ over $R$} and is denoted by $R(S)$.

Denoting by $z$ the zero of $S$, we have that $Z = \{\alpha z : \alpha \in R\}$ is an ideal of $R(S)$; the quotient $R_0(S) = R(S)/Z$ is called the \textit{contracted semigroup ring of $S$ over $R$}.
Each nonzero element $r \in R_0(S)$ can be expressed uniquely in the form

$$\sum_{i=1}^{n} \alpha_i x_i$$

for some $n \in \mathbb{N}$, some distinct elements $x_1, \ldots, x_n \in S \setminus \{0\}$, and some $\alpha_1, \ldots, \alpha_n \in R \setminus \{0\}$.

The set $\{x_1, \ldots, x_n\}$ is called the support of $r$ and is denoted by $\text{supp}(r)$; the elements $\alpha_1, \ldots, \alpha_n$ are the coefficients of $r$.

Since $R(S) \cong R_0(S^0)$, in case $S$ does not originally come with a zero element and one is adjoined to it, there is no loss in assuming that $S = S^0$. 

**Contracted semigroup ring**
Munn’s results

Munn studied (semi)primeness and (semi)primitivity of $R_0(S)$ for semigroups $S$ satisfying the following condition (eg: inverse semigroups)

**Condition (I)**

For every nonzero ideal $A$ of $R_0(S)$, there exists $a \in A \setminus 0$ and $e \in E_S \setminus 0$ such that $e \in \text{supp}(a) \subseteq H_e \cup (eSe \setminus (R_e \cap eSe))$. 
Theorem (Munn, 90)

R ring with identity, $S = S^0$ semigroup satisfying $(I)$
If $R(G)$ is semiprime (respectively, semiprimitive) for each nonzero maximal subgroup $G$ of $S$, then $R_0(S)$ is semiprime (respectively, semiprimitive).

Theorem (Munn, 90)

R ring with identity, $S = S^0$ regular semigroup satisfying $(I)$
If $S$ is 0-bisimple and $R(G)$ is prime (respectively, primitive) for some (every) nonzero maximal subgroup $G$ of $S$, then $R_0(S)$ is prime (respectively, primitive).
Partial converses

Even for inverse semigroups, all converses are false.

However, necessary conditions can be obtained, if a certain finiteness condition (introduced by Teply, Turman and Quesada) is imposed on the set of idempotents of $S$. 
Finiteness conditions

Let $E$ be a semilattice ($e^2 = e$, $ef = fe$, for all $e, f \in E$).

Recall that the natural partial order on $E$ is defined by $e \leq f$ if and only if $e = ef = fe$, for all $e, f \in E$.

For all $e, f \in E$, we say that $e$ covers $f$, and write $f \prec e$, if $f < e$ and, for all $g \in E$, the condition $f \leq g \leq e$ implies that either $g = f$ or $g = e$.

For $e \in E$, denote by $\hat{e}$ the set of elements covered by $e$.

We say that $E$ is pseudofinite if the following two conditions are satisfied:

(i) $\hat{e}$ is finite (possibly empty), for each $e \in E$;

(ii) for all $e, f \in E$, if $f < e$ then there exists $g \in E$ such that $f \leq g \prec e$. 
Munn’s results

Theorem (Munn, 87)

$R$ ring with identity, $S = S^0$ inverse semigroup such that $E_S$ is pseudofinite
Then $R(G)$ is semiprime (respectively, semiprimitive), for each nonzero maximal subgroup $G$ of $S$, if and only if $R_0(S)$ is semiprime (respectively, semiprimitive).

Theorem (Munn, 87)

$R$ ring with identity, $S = S^0$ inverse semigroup such that $E_S$ is pseudofinite
Then $S$ is 0-bisimple and $R(G)$ is prime (respectively, primitive), for each nonzero maximal subgroup $G$ of $S$, if and only if $R_0(S)$ is prime (respectively, primitive).
Recent results

Munn tried to generalize these results to other classes of semigroups (private communication to G.M.S. Gomes).

Theorem (Guo and Chen, 2012)

$R$ commutative ring with identity, $S$ finite ample semigroup

Then $R(S)$ is semiprimitive if and only if

(i) $S$ is an inverse semigroup

(ii) for all maximal subgroups $G$ of $S$, $R(G)$ is semiprimitive.

Note that:

- Inverse semigroups are ample.
- The regular elements of an ample semigroup form an inverse subsemigroup.

Ample semigroups have been studied extensively (Fountain, Lawson, ...).
Generalized Green’s relations - $R^*$ and $L^*$

Let $S$ be a semigroup. Consider the equivalence relations on $S$:

$$aR^* b \iff (\forall x, y \in S, \ xa = ya \iff xb = yb)$$

and, dually,

$$aL^* b \iff (\forall x, y \in S, \ ax = ay \iff bx = by) .$$

Clearly $R^*$ is a left congruence and $L^*$ is a right congruence.

Also consider the relations $H^* = R^* \cap L^*$ and $D^* = R^* \lor L^*$.

Note that $R^*$ and $L^*$ are generalizations of the familiar Green relations $R$ and $L$. In fact, $aR^* b$ if and only if $aR b$ in some oversemigroup of $S$, and dually for $L^*$. 
Ample semigroups

If each $\mathcal{L}^*$-class contains exactly one idempotent (denoted $a^*$ in $L_a^*$), we say that $S$ satisfies the “right ample condition” if:

$$(\text{AR}) \quad \forall a \in S, e \in E_S \quad ea = a(ea)^*.$$ 

Dually, if each $\mathcal{R}^*$-class contains exactly one idempotent (denoted $a^+$ in $R_a^*$), we say that $S$ satisfies the “left ample condition” if:

$$(\text{AL}) \quad \forall a \in S, e \in E_S \quad ae = (ae)^+ a.$$ 

The semigroup $S$ is said to be ample if $E_S$ is a semilattice (i.e., idempotents commute), each $\mathcal{R}^*$-class and each $\mathcal{L}^*$-class contain a unique idempotent and both the ample conditions (AR) and (AL) are satisfied.
Generalized Green’s relations - $\tilde{R}_E$ and $\tilde{L}_E$

Let $S$ be a semigroup, $E_S$ its set of idempotents, and $\text{Reg}(S)$ the set of regular elements in $S$; let $E \subseteq E_S$.

Consider the equivalence relations $\tilde{R}_E$ and $\tilde{L}_E$ defined by: for all $a, b \in S$,
$$a \tilde{R}_E b \iff \forall e \in E, \ ea = a \iff eb = b$$
and, dually,
$$a \tilde{L}_E b \iff \forall e \in E, \ ae = a \iff be = b$$

Consider also the equivalence relations $\tilde{H}_E = \tilde{R}_E \cap \tilde{L}_E$ and $\tilde{D}_E = \tilde{R}_E \lor \tilde{L}_E$. 
We say that $S$ is $\sim$-bisimple if it has a single $\tilde{D}_E$-class.

In case $S$ has a zero element, we say that $S$ is $0$-$\sim$-bisimple if it has a single nonzero $\tilde{D}_E$-class, that is, if $S/\tilde{D}_E = \{0, S \setminus 0\}$.

We have $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \tilde{\mathcal{R}}_E$ and $aRb$ whenever $a\tilde{R}_Eb$ with $a, b \in \text{Reg}(S)$. And dually for $\mathcal{L}$. 
Restriction semigroups with an inverse skeleton

*S is left restriction with distinguished semilattice* $E$ *if* $E$ *is a semilattice, the relation* $\tilde{R}_E$ *is a left congruence, each* $\tilde{R}_E$-*class contains a (necessarily unique) idempotent from* $E$ *and the left ample condition (AL) holds. Right restriction semigroups are defined dually.

*S is restriction* if it is left and right restriction with respect to the same distinguished semilattice $E$. In case $E = E_S$, we say that $S$ is a *weakly ample semigroup*.

Every inverse semigroup is ample and every ample semigroup is restriction with respect to $E_S$.

A restriction semigroup $S$ with distinguished semilattice $E$ has an *inverse $E$-skeleton* if each $\tilde{H}_E$-class $\tilde{H}_a$ contains a regular element $u$ for which there exists $u' \in V(u)$ such that $uu', u'u \in E$. In this case, each $\tilde{H}_E$-class of $S$ contains an element $u$ which has a unique inverse, say $u^{-1}$, such that $uu^{-1}, u^{-1}u \in E$. 
Analogue of Munn’s Condition (I)

The appropriate analogue of Munn’s condition holds for rings over restriction semigroups with an inverse skeleton.

Lemma
Let $R$ be a ring with identity, $S = S^0$ a restriction semigroup with an inverse $E$-skeleton, and $A$ a nonzero ideal of $R_0(S)$. Then there exists $e \in E \setminus 0$ and $a \in A \setminus 0$ such that

(i) $\text{supp}(a) \subseteq \tilde{H}_e \cup (eSe \setminus \tilde{R}_e)$;
(ii) $\text{supp}(a) \cap \tilde{H}_e \neq \emptyset$. 
Our results

**Theorem**

Let $R$ be a ring with identity and $S = S^0$ a restriction semigroup with an inverse $E$-skeleton. If $R(M)$ is semiprimitive (resp., semiprime) for each nonzero maximal reduced $(2, 1, 1)$-submonoid $M$ of $S$, then $R_0(S)$ is semiprimitive (resp., semiprime).

**Theorem**

Let $R$ be a ring with identity and $S = S^0$ a $0\sim$-bisimple restriction semigroup with an inverse $E$-skeleton. If $R(M)$ is primitive (resp., prime) for some nonzero maximal reduced $(2, 1, 1)$-submonoid $M$ of $S$, then $R_0(S)$ is primitive (resp., prime).
Remarks

A restriction semigroup can be seen as a $(2,1,1)$-algebra with respect to the operations $\cdot$, $+$, and $\ast$.

A $(2,1,1)$-submonoid $M$ of a restriction semigroup $S$ with distinguished semilattice $E$ is a $(2,1,1)$-subalgebra of $S$ which is a monoid, and is thus restriction with distinguished semilattice $E' = E \cap E_M$.

By a reduced restriction semigroup we mean a monoid $M$ with identity $1_M$ viewed as a restriction semigroup with distinguished semilattice $E = \{1_M\}$. Note that $x^+ = x^* = 1_M$, for all $x \in M$.

Clearly, any cancellative monoid is unipotent and any unipotent $(2,1,1)$-monoid is reduced.
Lemma

Let $S$ be a restriction semigroup with distinguished semilattice $E$. Then the maximal reduced \((2, 1, 1)\)-submonoids of $S$ are precisely the $\tilde{\mathcal{H}}_E$-classes $\tilde{\mathcal{H}}_e$ with $e \in E$. If $S$ is weakly ample (respectively, ample), they are the maximal unipotent (respectively, cancellative) \((2, 1, 1)\)-submonoids.

The primeness and semiprimeness of the rings $R(M)$, for a cancellative monoid $M$, were studied by Okniński (93) and Clase (98); the semiprimitivity was studied by Okniński (94).

The question regarding algebras over reduced restriction and unipotent semigroups is open.
Our results - pseudofinite case

Theorem
Let \( S = S^0 \) be a restriction semigroup with an inverse \( E \)-skeleton such that \( E \) is pseudofinite. Let \( R \) be a ring with identity. Then \( R_0(S) \) is semiprimitive (resp., semiprime) if and only if \( R(M) \) is semiprimitive (resp., semiprime) for each nonzero maximal reduced \((2, 1, 1)\)-submonoid \( M \) of \( S \).

Theorem
Let \( S = S^0 \) be a restriction semigroup with an inverse \( E \)-skeleton such that \( E \) is pseudofinite. Let \( R \) be a ring with identity. Then \( R_0(S) \) is primitive (resp., prime) if and only if \( S \) is \( 0\sim \)-bisimple and \( R(M) \) is primitive (resp., prime) for some (each) nonzero maximal reduced \((2, 1, 1)\)-submonoid \( M \) of \( S \).
Rukolaњe idempotents

*R ring with identity, $S = S^0$ semigroup such that $E_S$ is a pseudofinite semilattice

The *Rukolaњe idempotents* are defined, for each $e \in E \setminus \{0\}$, as the (finite) product of all the (commuting) factors $e - g$, where $g \in E$ is covered by $e$:

$$\sigma(e) = \prod_{g \in \hat{e}} (e - g).$$

Note that $\hat{e} \neq \emptyset$, for all $e \in E \setminus \{0\}$, since $S$ has a zero element.

**Lemma**

*Let* $S = S^0$ *be a semigroup such that* $E_S$ *is a pseudofinite semilattice. Then:*

(i) for each $e \in E_S \setminus \{0\}$, $\sigma(e)$ is a nonzero idempotent of $R_0(S)$ such that $e\sigma(e) = \sigma(e) = \sigma(e)e$.

(ii) for all $e, f \in E_S \setminus \{0\}$ with $e \neq f$, $\sigma(e)\sigma(f) = 0$. 
Assume:

\( S = S^0 \) is a restriction semigroup with an inverse \( E \)-skeleton.

Fix \( e \in E \) and consider \( \tilde{D} = \tilde{D}_e \).

For each \( f \in E_{\tilde{D}} \), there exists a regular element \( t_f \in S \) such that \( e \tilde{R}_E t_f \tilde{L}_E f \) and for which its (unique) inverse \( t_f^{-1} \) is such that \( t_f t_f^{-1}, t_f^{-1} t_f \in E \).

Fix a transversal \( T = \{t_f \in T_{e,f} : f \in E_{\tilde{D}}\} \).
Proposition

Let $S = S^0$ be a restriction semigroup with an inverse $E$-skeleton such that $E$ is pseudofinite. Let $e \in E$ and $\tilde{D} = \tilde{D}_e$. Let $R$ be a ring with identity, $K$ be a two-sided ideal of $R(\tilde{H}_e)$ and consider

$$M(K) = \sum_{f,g \in E_{\tilde{D}}} \sigma(f) t_f^{-1} K t_g \sigma(g).$$

Then

(i) $M(K)$ is a two-sided ideal of $R_0(S)$.

(ii) $M(K)$ is isomorphic to the ring $\mathcal{M}_{|E_{\tilde{D}}|}(K)$ of all $|E_{\tilde{D}}| \times |E_{\tilde{D}}|$-matrices over $K$ with at most finitely many nonzero entries.
Sketch-proof for semiprimitivity in the pseudofinite case

Suppose $R_0(S)$ is semiprimitive and let $e \in E_S$.

Since $K = J(R(H_e^*))$ is a two-sided ideal of $R(H_e^*)$, we can consider the ideal $M(K)$ of $R_0(S)$, which we know to be isomorphic to the ring $\mathcal{M}_\nu(K)$, where $\nu = |E_D^*|$.

Then

$$M(K) \cong \mathcal{M}_\nu(K) = \mathcal{M}_\nu(J(K)) = J(\mathcal{M}_\nu(K)) \cong J(M(K))$$

Therefore, $M(K) \subseteq J(R_0(S)) = 0$ and so $M(K) = 0$.

Thus, $\mathcal{M}_\nu(K) = 0$ and, hence, $K = 0$, that is, $J(R(\tilde{H}_e)) = 0$.

Hence, $R(H_e^*)$ is semiprimitive.
Questions

1. If $M$ is a cancellative monoid, when is $R(M)$ primitive?

2. If $M$ is a unipotent monoid (or reduced restriction), what can be said about $R(M)$?
Free restriction semigroup

The behaviour of the free restriction semigroup is entirely different from its inverse analogue, although the free restriction semigroup (on a set $X$) is a subsemigroup of the free inverse semigroup $FIS_X$ on $X$ and both share the same set of idempotents.

The free restriction semigroup on a set $X$ coincides with the free ample semigroup $FAS_X$ on a set $X$ and cannot have an inverse skeleton.

The algebra of $FIS_X$ is always semiprimitive, and thus always semiprime, and is prime iff primitive iff $X$ is infinite.

Guo and Shum claim that the semigroup algebra of $FAS_X$ is not semiprime, regardless of the finitude of $X$ — and, therefore, it is neither prime, nor semiprimitive, nor primitive.
Examples

Let $M$ be a monoid, $I$ a set, and consider the Brandt monoid $S = B(M, I) = (I \times M \times I) \cup \{0\}$, where all products involving 0 yield 0 and $(i, a, j)(k, b, l) = (i, ab, l)$ if $j = k$ and 0 otherwise.

Then, denoting by 1 the identity of $M$, we have that $S$ is a restriction semigroup with distinguished semilattice $E = \{(i, 1, i) : i \in I\} \cup \{0\} \subseteq E_S$, where $(i, a, j)^+ = (i, 1, i)$ and $(i, a, j)^* = (j, 1, j)$, for all $(i, a, j) \in S \setminus 0$.

Clearly, in case $M$ is unipotent, $E = E_S$ and $S$ is an example of a weakly ample semigroup.

In either case, the $\tilde{H}_E$-class of an arbitrary nonzero element $(i, a, j)$ consists of all elements $(i, b, j)$ with $b \in M$, and, in particular, $(i, 1, j) \in \tilde{H}_{(i, a, j)} \cap \text{Reg}(S)$, with inverse $(j, 1, i)$ and their product in $E$.

Therefore, $S$ has an inverse $E$-skeleton.

That $E$ is pseudofinite follows trivially from the fact that, restricted to $E$, the natural partial order reduces to equality (that is, $E$ is an anti-chain).
Examples

Let $M$ be a monoid and $\theta: M \to M$ a morphism from $M$ into its group of units and consider the Bruck-Reilly extension of $M$ determined by $\theta$, that is, the monoid $BR(M, \theta) = \mathbb{Z} \times M \times \mathbb{Z}$ (where $\mathbb{Z}$ denotes the non-negative integers) equipped with the operation $(m, a, n)(p, b, q) = (m - n + t, a\theta^{t-n} b\theta^{t-p}, q - p + t)$ with $t = \max\{n, p\}$, where $\theta^0 = id_M$.

Then $S = (BR(M, \theta))^0$ is a restriction semigroup with distinguished semilattice $E = \{(m, 1, m): m \in \mathbb{Z}\} \cup \{0\}$.

Similarly to the previous example, we have $(m, a, n)^+ = (m, 1, m)$ and $(m, a, n)^* = (n, 1, n)$, for all $(m, a, n) \in S$, and, thus, $(m, 1, n) \in H_{(m,a,n)}^E \cap \text{Reg}(S)$.

This turn, $E$ is pseudofinite because it consists of a chain, as it can be straightforwardly checked.
References


