Semigroup algebra of a restriction semigroup with an inverse skeleton

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Starting point

R ring, G group, $R(G)$ group ring

Well studied (Connell, Passman, ...):

- $R(G)$ prime, semiprime
- $R(G)$ primitive, semiprimitive

Theorem (Domanov, 76)

F field, S inverse semigroup
If $F(G)$ is semiprimitive for every nonzero maximal subgroup G of S, then $F(S)$ is semiprimitive.

Theorem (Domanov, 76)

F field, S inverse semigroup
If S is 0-bisimple and $F(G)$ is primitive for every nonzero maximal subgroup G of S, then $F(S)$ is primitive.

Converse false (Teply, Turman and Quesada, 80).

Primeness and Semiprimeness

A ring, not necessarily with identity

The ring A is *prime* if for all (left, right, two-sided) ideals I and J of A such that $IJ = 0$, then either $I = 0$ or $J = 0$.

The ring A is *semiprime* if for any (left, right, two-sided) ideal I of A such that $I^2 = 0$, then $I = 0$.

Primitivity and Semiprimitivity

M right A-module

The set $(0\colon M) = \{\, a \in A\colon\ Ma = 0 \,\}$ is called the *(right) annihilator of* M and is an ideal of A.

M is *faithful* if $(0\colon M) = 0$.

M is *simple* if $M \neq 0$ and M has no proper submodules.

M is *semisimple* if it is the direct sum of simple submodules.

The ring A is *right primitive* if it admits a simple faithful right module.

The ring A is *semiprimitive* if it admits a semisimple faithful right module.

Remarks

- Semiprimitivity is a left-right symmetric concept.
- Primitivity is not left-right symmetric.
- Every primitive ring is prime and semiprimitive.
- Both prime and semiprimitive rings are semiprime.

Jacobson radical

An element $a \in A$ is *left quasiregular* if there exists $r \in A$ such that $r + a + ra = 0$.

A (left, right or two-sided) ideal I of A is said to be *left quasiregular* if every element of I is left quasiregular.

Right quasiregular elements and right quasiregular ideals are defined analogously.

The *Jacobson radical* $J(A)$ of A can be characterized as the (left, right) quasiregular (left, right) ideal of A which contains every (left, right) quasiregular ideal.

Recall: A is semiprimitive if and only if $J(A) = 0$.

Contracted semigroup ring

S semigroup with zero, R ring with identity

The set of finite formal sums

$$\sum_{x \in S} \alpha_x x$$

with coefficients in R, equipped with the obvious definition of addition and multiplication, is the *semigroup ring of S over R* and is denoted by $R(S)$.

Denoting by z the zero of S, we have that $Z = \{\alpha z \colon \alpha \in R\}$ is an ideal of $R(S)$; the quotient $R_0(S) = R(S)/Z$ is called the *contracted semigroup ring of S over R*.

Contracted semigroup ring

Each nonzero element $r \in R_0(S)$ can be expressed uniquely in the form

$$\sum_{i=1}^{n} \alpha_i x_i$$

for some $n \in \mathbb{N}$, some distinct elements $x_1, \ldots, x_n \in S \setminus \{0\}$, and some $\alpha_1, \ldots, \alpha_n \in R \setminus \{0\}$.

The set $\{x_1, \ldots, x_n\}$ is called the *support of* r and is denoted by $\operatorname{supp}(r)$; the elements $\alpha_1, \ldots, \alpha_n$ are the *coefficients of* r.

Since $R(S) \simeq R_0(S^0)$, in case S does not originally come with a zero element and one is adjoined to it, there is no loss in assuming that $S = S^0$.

Munn's results

Munn studied (semi)primeness and (semi)primitivity of $R_0(S)$ for semigroups S satisfying the following condition (eg: inverse semigroups)

Condition (I)

For every nonzero ideal A of $R_0(S)$, there exists $a \in A \setminus 0$ and $e \in E_S \setminus 0$ such that $e \in supp(a) \subseteq H_e \cup (eSe \setminus (R_e \cap eSe))$.

Munn's results

Theorem (Munn, 90)

R ring with identity, $S = S^0$ semigroup satisfying (I)
If $R(G)$ is semiprime (respectively, semiprimitive) for each nonzero maximal subgroup G of S, then $R_0(S)$ is semiprime (respectively, semiprimitive).

Theorem (Munn, 90)

R ring with identity, $S = S^0$ regular semigroup satisfying (I)
If S is 0-bisimple and $R(G)$ is prime (respectively, primitive) for some (every) nonzero maximal subgroup G of S, then $R_0(S)$ is prime (respectively, primitive).

Partial converses

Even for inverse semigroups, all converses are false.

However, necessary conditions can be obtained, if a certain finiteness condition (introduced by Teply, Turman and Quesada) is imposed on the set of idempotents of S.

Finiteness conditions

Let E be a semilattice ($e^2 = e$, $ef = fe$, for all $e, f \in E$).

Recall that the *natural partial order on E* is defined by $e \leqslant f$ if and only if $e = ef = fe$, for all $e, f \in E$.

For all $e, f \in E$, we say that *e covers f*, and write $f \prec e$, if $f < e$ and, for all $g \in E$, the condition $f \leqslant g \leqslant e$ implies that either $g = f$ or $g = e$.

For $e \in E$, denote by $\hat{e}$ the set of elements covered by e.

We say that E is *pseudofinite* if the following two conditions are satisfied:

(i) $\hat{e}$ is finite (possibly empty), for each $e \in E$;

(ii) for all $e, f \in E$, if $f < e$ then there exists $g \in E$ such that $f \leqslant g \prec e$.

Munn's results

Theorem (Munn, 87)

R ring with identity, $S = S^0$ inverse semigroup such that E_S is pseudofinite
Then $R(G)$ is semiprime (respectively, semiprimitive), for each nonzero maximal subgroup G of S, if and only if $R_0(S)$ is semiprime (respectively, semiprimitive).

Theorem (Munn, 87)

R ring with identity, $S = S^0$ inverse semigroup such that E_S is pseudofinite
Then S is 0-bisimple and $R(G)$ is prime (respectively, primitive), for each nonzero maximal subgroup G of S, if and only if $R_0(S)$ is prime (respectively, primitive).

Recent results

Munn tried to generalize these results to other classes of semigroups (private communication to G.M.S. Gomes).

Theorem (Guo and Chen, 2012)

R commutative ring with identity, S finite ample semigroup
Then $R(S)$ is semiprimitive if and only if

(i) *S is an inverse semigroup*

(ii) *for all maximal subgroups G of S, $R(G)$ is semiprimitive.*

Note that:

- Inverse semigroups are ample.
- The regular elements of an ample semigroup form an inverse subsemigroup.

Ample semigroups have been studied extensively (Fountain, Lawson, ...).

Generalized Green's relations - $\mathcal{R}^*$ and $\mathcal{L}^*$

Let S be a semigroup. Consider the equivalence relations on S:

$$a\mathcal{R}^*b \iff (\forall\, x, y \in S, \quad xa = ya \Leftrightarrow xb = yb)$$

and, dually,

$$a\mathcal{L}^*b \iff (\forall\, x, y \in S, \quad ax = ay \Leftrightarrow bx = by)\ .$$

Clearly $\mathcal{R}^*$ is a left congruence and $\mathcal{L}^*$ is a right congruence.

Also consider the relations $\mathcal{H}^* = \mathcal{R}^* \cap \mathcal{L}^*$ and $\mathcal{D}^* = \mathcal{R}^* \vee \mathcal{L}^*$.

Note that $\mathcal{R}^*$ and $\mathcal{L}^*$ are generalizations of the familiar Green relations $\mathcal{R}$ and $\mathcal{L}$. In fact, $a\mathcal{R}^*b$ if and only if $a\mathcal{R}b$ in some oversemigroup of S, and dually for $\mathcal{L}^*$.

Ample semigroups

If each $\mathcal{L}^*$-class contains exactly one idempotent (denoted a^* in L_a^*), we say that S satisfies the "right ample condition" if:

$$\text{(AR)} \;\; \forall\, a \in S, e \in E_S \quad ea = a(ea)^*.$$

Dually, if each $\mathcal{R}^*$-class contains exactly one idempotent (denoted a^+ in R_a^*), we say that S satisfies the "left ample condition" if:

$$\text{(AL)} \;\; \forall\, a \in S, e \in E_S \quad ae = (ae)^+ a.$$

The semigroup S is said to be *ample* if E_S is a semilattice (i.e., idempotents commute), each $\mathcal{R}^*$-class and each $\mathcal{L}^*$-class contain a unique idempotent and both the ample conditions (AR) and (AL) are satisfied.

Generalized Green's relations - $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$

Let S be a semigroup, E_S its set of idempotents, and $\mathrm{Reg}(S)$ the set of regular elements in S; let $E \subseteq E_S$.

Consider the equivalence relations $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$ defined by: for all $a, b \in S$,

$$a \widetilde{\mathcal{R}}_E b \iff \forall\, e \in E\,,\ ea = a \Leftrightarrow eb = b$$

and, dually,

$$a \widetilde{\mathcal{L}}_E b \iff \forall\, e \in E\,,\ ae = a \Leftrightarrow be = b\,.$$

Consider also the equivalence relations $\widetilde{\mathcal{H}}_E = \widetilde{\mathcal{R}}_E \cap \widetilde{\mathcal{L}}_E$ and $\widetilde{\mathcal{D}}_E = \widetilde{\mathcal{R}}_E \vee \widetilde{\mathcal{L}}_E$.

Remarks

We say that S is $\sim$-*bisimple* if it has a single $\widetilde{\mathcal{D}}_E$-class.

In case S has a zero element, we say that S is 0-$\sim$-*bisimple* if it has a single nonzero $\widetilde{\mathcal{D}}_E$-class, that is, if $S/\widetilde{\mathcal{D}}_E = \{0, S \setminus 0\}$.

We have $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \widetilde{\mathcal{R}}_E$ and $a\mathcal{R}b$ whenever $a\widetilde{\mathcal{R}}_E b$ with $a, b \in \mathrm{Reg}(S)$. And dually for $\mathcal{L}$.

Restriction semigroups with an inverse skeleton

S is *left restriction with distinguished semilattice* E if E is a semilattice, the relation $\widetilde{\mathcal{R}}_E$ is a left congruence, each $\widetilde{\mathcal{R}}_E$-class contains a (necessarily unique) idempotent from E and the left ample condition (AL) holds. *Right restriction semigroups* are defined dually.

S is *restriction* if it is left and right restriction with respect to the same distinguished semilattice E. In case $E = E_S$, we say that S is a *weakly ample semigroup*.

Every inverse semigroup is ample and every ample semigroup is restriction with respect to E_S.

A restriction semigroup S with distinguished semilattice E has an *inverse E-skeleton* if each $\widetilde{\mathcal{H}}_E$-class $\widetilde{H}_a$ contains a regular element u for which there exists $u' \in V(u)$ such that $uu', u'u \in E$. In this case, each $\widetilde{\mathcal{H}}_E$-class of S contains an element u which has a unique inverse, say u^{-1}, such that $uu^{-1}, u^{-1}u \in E$.

Analogue of Munn's Condition (I)

The appropriate analogue of Munn's condition holds for rings over restriction semigroups with an inverse skeleton.

Lemma

Let R be a ring with identity, $S = S^0$ a restriction semigroup with an inverse E-skeleton, and A a nonzero ideal of $R_0(S)$. Then there exists $e \in E \setminus 0$ and $a \in A \setminus 0$ such that

(i) $supp(a) \subseteq \widetilde{H}_e \cup (eSe \setminus \widetilde{R}_e)$;

(ii) $supp(a) \cap \widetilde{H}_e \neq \varnothing$.

Our results

Theorem
Let R be a ring with identity and $S = S^0$ a restriction semigroup with an inverse E-skeleton. If $R(M)$ is semiprimitive (resp., semiprime) for each nonzero maximal reduced $(2, 1, 1)$-submonoid M of S, then $R_0(S)$ is semiprimitive (resp., semiprime).

Theorem
Let R be a ring with identity and $S = S^0$ a 0-$\sim$-bisimple restriction semigroup with an inverse E-skeleton. If $R(M)$ is primitive (resp., prime) for some nonzero maximal reduced $(2, 1, 1)$-submonoid M of S, then $R_0(S)$ is primitive (resp., prime).

Remarks

A restriction semigroup can be seen as a $(2, 1, 1)$-algebra with respect to the operations $\cdot$, $^+$, and *.

A $(2, 1, 1)$-*submonoid* M of a restriction semigroup S with distinguished semilattice E is a $(2, 1, 1)$-subalgebra of S which is a monoid, and is thus restriction with distinguished semilattice $E' = E \cap E_M$.

By a *reduced restriction* semigroup we mean a monoid M with identity 1_M viewed as a restriction semigroup with distinguished semilattice $E = \{1_M\}$. Note that $x^+ = x^* = 1_M$, for all $x \in M$.

Clearly, any cancellative monoid is unipotent and any unipotent $(2, 1, 1)$-monoid is reduced.

Remarks

Lemma

Let S be a restriction semigroup with distinguished semilattice E. Then the maximal reduced $(2,1,1)$-submonoids of S are precisely the $\widetilde{\mathcal{H}}_E$-classes $\widetilde{H}_e$ with $e \in E$. If S is weakly ample (respectively, ample), they are the maximal unipotent (respectively, cancellative) $(2,1,1)$-submonoids.

The primeness and semiprimeness of the rings $R(M)$, for a cancellative monoid M, were studied by Okniński (93) and Clase (98); the semiprimitivity was studied by Okniński (94).

The question regarding algebras over reduced restriction and unipotent semigroups is open.

Our results - pseudofinite case

Theorem

Let $S = S^0$ be a restriction semigroup with an inverse E-skeleton such that E is pseudofinite. Let R be a ring with identity. Then $R_0(S)$ is semiprimitive (resp., semiprime) if an only if $R(M)$ is semiprimitive (resp., semiprime) for each nonzero maximal reduced $(2, 1, 1)$-submonoid M of S.

Theorem

Let $S = S^0$ be a restriction semigroup with an inverse E-skeleton such that E is pseudofinite. Let R be a ring with identity. Then $R_0(S)$ is primitive (resp., prime) if and only if S is 0-$\sim$-bisimple and $R(M)$ is primitive (resp., prime) for some (each) nonzero maximal reduced $(2, 1, 1)$-submonoid M of S.

Rukolaĭne idempotents

R ring with identity, $S = S^0$ semigroup such that E_S is a pseudofinite semilattice

The *Rukolaĭne idempotents* are defined, for each $e \in E \setminus \{0\}$, as the (finite) product of all the (commuting) factors $e - g$, where $g \in E$ is covered by e:

$$\sigma(e) = \prod_{g \in \hat{e}} (e - g)\,.$$

Note that $\hat{e} \neq \emptyset$, for all $e \in E \setminus \{0\}$, since S has a zero element.

Lemma

Let $S = S^0$ be a semigroup such that E_S is a pseudofinite semilattice. Then:

(i) *for each $e \in E_S \setminus \{0\}$, $\sigma(e)$ is a nonzero idempotent of $R_0(S)$ such that $e\sigma(e) = \sigma(e) = \sigma(e)e$.*

(ii) *for all $e, f \in E_S \setminus \{0\}$ with $e \neq f$, $\sigma(e)\sigma(f) = 0$.*

Ideals

Assume:

$S = S^0$ is a restriction semigroup with an inverse E-skeleton.

Fix $e \in E$ and consider $\widetilde{D} = \widetilde{D}_e$.

For each $f \in E_{\widetilde{D}}$, there exists a regular element $t_f \in S$ such that $e \widetilde{\mathcal{R}}_E t_f \widetilde{\mathcal{L}}_E f$ and for which its (unique) inverse t_f^{-1} is such that $t_f t_f^{-1}, t_f^{-1} t_f \in E$.

Fix a transversal $T = \{t_f \in T_{e,f} \colon f \in E_{\widetilde{D}}\}$.

Ideals

Proposition

Let $S = S^0$ be a restriction semigroup with an inverse E-skeleton such that E is pseudofinite. Let $e \in E$ and $\widetilde{D} = \widetilde{D}_e$. Let R be a ring with identity, K be a two-sided ideal of $R(\widetilde{H}_e)$ and consider

$$M(K) = \sum_{f,g \in E_{\widetilde{D}}} \sigma(f) t_f^{-1} K t_g \sigma(g)\,.$$

Then

(i) *$M(K)$ is a two-sided ideal of $R_0(S)$.*

(ii) *$M(K)$ is isomorphic to the ring $\mathcal{M}_{|E_{\widetilde{D}}|}(K)$ of all $|E_{\widetilde{D}}| \times |E_{\widetilde{D}}|$-matrices over K with at most finitely many nonzero entries.*

Sketch-proof for semiprimitivity in the pseudofinite case

Suppose $R_0(S)$ is semiprimitive and let $e \in E_S$.

Since $K = J(R(H_e^*))$ is a two-sided ideal of $R(H_e^*)$, we can consider the ideal $M(K)$ of $R_0(S)$, which we know to be isomorphic to the ring $\mathcal{M}_\nu(K)$, where $\nu = |E_{D^*}|$.

Then

$$M(K) \simeq \mathcal{M}_\nu(K) = \mathcal{M}_\nu(J(K)) = J(\mathcal{M}_\nu(K)) \simeq J(M(K))$$

Therefore, $M(K) \subseteq J(R_0(S)) = 0$ and so $M(K) = 0$.

Thus, $\mathcal{M}_\nu(K) = 0$ and, hence, $K = 0$, that is, $J(R(\widetilde{H}_e)) = 0$.

Hence, $R(H_e^*)$ is semiprimitive.

Questions

1. If M is a cancellative monoid, when is $R(M)$ primitive?
2. If M is a unipotent monoid (or reduced restriction), what can be said about $R(M)$?

Free restriction semigroup

The behaviour of the free restriction semigroup is entirely different from its inverse analogue, although the free restriction semigroup (on a set X) is a subsemigroup of the free inverse semigroup FIS_X on X and both share the same set of idempotents.

The free restriction semigroup on a set X coincides with the free ample semigroup FAS_X on a set X and cannot have an inverse skeleton.

The algebra of FIS_X is always semiprimitive, and thus always semiprime, and is prime iff primitive iff X is infinite.

Guo and Shum claim that the semigroup algebra of FAS_X is not semiprime, regardless of the finitude of X — and, therefore, it is neither prime, nor semiprimitive, nor primitive.

Examples

Let M be a monoid, I a set, and consider the Brandt monoid $S = B(M, I) = (I \times M \times I) \cup \{0\}$, where all products involving 0 yield 0 and $(i, a, j)(k, b, l) = (i, ab, l)$ if $j = k$ and 0 otherwise.
Then, denoting by 1 the identity of M, we have that S is a restriction semigroup with distinguished semilattice $E = \{(i, 1, i) \colon i \in I\} \cup \{0\} \subseteq E_S$, where $(i, a, j)^+ = (i, 1, i)$ and $(i, a, j)^* = (j, 1, j)$, for all $(i, a, j) \in S \setminus 0$.
Clearly, in case M is unipotent, $E = E_S$ and S is an example of a weakly ample semigroup.
In either case, the $\widetilde{\mathcal{H}}_E$-class of an arbitrary nonzero element (i, a, j) consists of all elements (i, b, j) with $b \in M$, and, in particular, $(i, 1, j) \in \widetilde{H}_{(i,a,j)} \cap \mathrm{Reg}(S)$, with inverse $(j, 1, i)$ and their product in E.
Therefore, S has an inverse E-skeleton.
That E is pseudofinite follows trivially from the fact that, restricted to E, the natural partial order reduces to equality (that is, E is an anti-chain).

Examples

Let M be a monoid and $\theta\colon M \to M$ a morphism from M into its group of units and consider the Bruck-Reilly extension of M determined by θ, that is, the monoid $BR(M,\theta) = \mathbb{Z} \times M \times \mathbb{Z}$ (where $\mathbb{Z}$ denotes the non-negative integers) equipped with the operation $(m,a,n)(p,b,q) = (m-n+t, a\theta^{t-n}\, b\theta^{t-p}, q-p+t)$ with $t = \max\{n,p\}$, where $\theta^0 = id_M$.
Then $S = (BR(M,\theta))^0$ is a restriction semigroup with distinguished semilattice $E = \{(m,1,m)\colon m \in \mathbb{Z}\} \cup \{0\}$.
Similarly to the previous example, we have $(m,a,n)^+ = (m,1,m)$ and $(m,a,n)^* = (n,1,n)$, for all $(m,a,n) \in S$, and, thus, $(m,1,n) \in \widetilde{H}^E_{(m,a,n)} \cap \mathrm{Reg}(S)$.
This turn, E is pseudofinite because it consists of a chain, as it can be straightforwardly checked.

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