Ideals in βS

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NBSAN, York, Wednesday 11 May 2016

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Semigroups and ideals

Let S be a semigroup.

Basic examples $(\mathbb{N}, +)$ and $(\mathbb{Z}, +)$; $\mathbb{F}_2$ = free group on 2 generators.

An element $p \in S$ is **idempotent** if $p^2 = p$; the set of these is $E(S)$. Set $p \leq q$ in $E(S)$ if $p = pq = qp$; $(E(S), \leq)$ is a partially ordered set, and we may have **minimal** idempotents.

For $s \in S$, set $L_s(t) = st$ and $R_s(t) = ts$ for $t \in S$. An element $s \in S$ is **cancellable** if both L_s and R_s are injective, and S is **cancellative** if each $s \in S$ is cancellable.

A subset $I \subset S$ is a **left ideal** if $sx \in I$ for each $s \in S$ and $x \in I$, i.e., $SI \subset I$. Similarly, we have a **right ideal**. An **ideal** is a subset that is both a left and right ideal. The minimum ideal (if it exists) is denoted by $K(S)$.

Stone–Čech compactifications

The **Stone–Čech compactification** of a set S is denoted by βS; we regard S as a subset of βS, and set $S^* = \beta S \setminus S$; this is the **growth** of S. Especially we consider $\beta\mathbb{N}$ and $\mathbb{N}^*$.

The space βS is each of the following:

• - abstractly characterized by a universal property: βS is a compactification of S such that each bounded function from S to a compact space K has an extension to a continuous map from βS to K;

• - the space of ultrafilters on S;

• - the Stone space of the Boolean algebra $\mathcal{P}(S)$, the power set of S;

• - the character space of the commutative C^*-algebra $\ell^\infty(S)$, so that $\ell^\infty(S) = C(\beta S)$. See below.

Some properties

The space βS is big: if $|S| = \kappa$, then $|\beta S| = 2^{2^\kappa}$. In particular, $|\beta\mathbb{N}| = 2^{\mathfrak{c}}$, which may be $\aleph_2$.

Topologically βS is a **Stonean space**: it is **extremely disconnected**, so that the closure of every open set is also open. (But S^* is not Stonean.)

Many questions about $(\mathbb{N}, +)$, including combinatorical questions, can be resolved by moving up to $\beta\mathbb{N}$ - see the talk of Dona Strauss.

Semigroup compactifications

Let S be a semigroup. Then βS becomes a semigroup, as follows.

For each $s \in S$, the map $L_s : S \to \beta S$ has an extension to a continuous map $L_s : \beta S \to \beta S$. For $u \in \beta S$, define $s \,\square\, u = L_s(u)$.

Next, the map $R_u : s \mapsto s \,\square\, u,\ S \to \beta S$, has an extension to a continuous map $R_u : \beta S \to \beta S$ for each $u \in \beta S$. Define

$$u \,\square\, v = R_v(u) \quad (u, v \in \beta S)\,.$$

Then $(\beta S, \square)$ is a compact, right topological semigroup (to be explained later).

Similarly $(\beta S, \diamond)$ is a compact, left topological semigroup.

Fact S^* is an ideal in $(\beta S, \square)$ whenever S is cancellative. $\square$

$\beta\mathbb{N}$

Often the binary operation on $\beta\mathbb{N}$ from the semigroup $(\mathbb{N}, +)$ is denoted by $+$ to give the semigroup $(\beta\mathbb{N}, +)$. But note that $x + y \neq y + x$, in general.

Example $\mathbb{N}^*$ is a closed left ideal in $(\beta\mathbb{Z}, \Box)$, but not a right ideal. $\Box$

There are many deep theorems about $(\beta\mathbb{N}, +)$ and $(\beta S, \Box)$ (see the book [HS] of **Hindman–Strauss**); many basic open questions remain.

Some answers may be independent of ZFC.

Compact, right topological semigroup

Definition A semigroup V with a topology τ is a **compact, right topological semigroup** if (V, τ) is a compact space and the map R_v is continuous with respect to τ for each $v \in V$.

For example, $V = (\beta S, \Box)$, or $(S^*, \Box)$ for cancellative S. It is the maximal such compactification.

In general, the maps L_v are not continuous on these semigroups. For example, let $V = (\beta S, \Box)$. Then L_v is continuous when $v \in S$; for cancellative semigroups S, L_v is continuous **only** when $v \in S$.

The structure theorem

Study of these semigroups is based on the following **structure theorem**; see [HS].

Theorem Let V be a compact, right topological semigroup.

(i) A unique minimum ideal $K(V)$ exists in V. The families of minimal left ideals and of minimal right ideals of V both partition $K(V)$.

(ii) For each minimal right and left ideals R and L in V, there exists an element $p \in E(V) \cap R \cap L$ such that $R \cap L = RL = pVp$ is a group; these groups are maximal in $K(V)$, are pairwise isomorphic, and the family of these groups partitions $K(V)$.

(iii) For each $p, q \in K(V)$, the subset $pK(V)q$ is a subgroup of V, and there exists $r \in E(K(V))$ with $rp = p$ and $qr = q$.

(iv) $E(V) \cap K(V) = \{\text{minimal idempotents}\}$. $\square$

$K(\beta\mathbb{N})$ is big

It is easy to see that $K(\beta\mathbb{N})$ is equal to $K(\mathbb{N}^*)$.

Theorem (**Hindman and Pym**) The semigroup $K(\mathbb{N}^*)$ contains a copy of the free semigroup on $2^{\mathfrak{c}}$ generators. $\square$

A semigroup R of the form $A \times B$, where

$$(a,b)(c,d) = (a,d) \quad (a,c \in A,\ b,d \in B)$$

is a **rectangular semigroup**. It is a deep result of **Yevhen Zelenyuk** that $K(\mathbb{N}^*)$ contains a rectangular semigroup $A \times B$ with $|A| = |B| = 2^{\mathfrak{c}}$. Thus there is a 'very large' sub-semigroup R of $K(\mathbb{N}^*)$.

Algebras

An **algebra** is linear space (over $\mathbb{C}$) that also has an associative product such that the distributive laws hold and the product is compatible with scalar multiplication.

Examples (1) $\mathbb{M}_n$ - this is $n \times n$ matrices over $\mathbb{C}$. It is called the **full matrix algebra**.

(2) Start with a semigroup S. Let δ_s denote the characteristic function of s. Define

$$\delta_s \star \delta_t = \delta_{st} .$$

Consider the finite sums of the δ_s with the obvious product. This is the (algebraic) **semigroup algebra**, $\mathbb{C}S = \mathrm{lin}\{\delta_s : s \in S\}$. $\square$

Ideals in algebras

Let A be an algebra. A **left ideal** is a linear subspace I such that $AI \subset I$. A **maximal left ideal** is a proper left ideal that is maximal with respect to inclusion.

The **radical** of A, called rad$A$, is the intersection of the maximal left ideals.

It is also equal to the intersection of the maximal right ideals, and so rad$A$ is an ideal in A.

The algebra is **semi-simple** if rad$A = \{0\}$. It is easy to see that $A/$rad$A$ is always a semi-simple algebra.

Banach spaces

Let E be a Banach space. Then a linear functional λ is **bounded** if

$$\|\lambda\| = \sup\{|\lambda(x)| : \|x\| \leq 1\} < \infty\,.$$

Write E' for the space of these bounded linear functionals; so $(E', \|\cdot\|)$ is a Banach space. It is the **dual space** of E.

Write $\langle x,\, \lambda\rangle$ for $\lambda(x)$. Thus $\langle\ ,\ \rangle$ gives the **duality**.

The **weak-$*$ topology** on E' is such that $\lambda_\alpha \to 0$ iff $\langle x,\, \lambda_\alpha\rangle \to 0$ for each $x \in E$. The closed unit ball of E' is weak-$*$ compact.

The **bidual** is $E'' = (E')'$. The map

$$\kappa : E \to E''\,,$$

where $\langle \kappa(x),\, \lambda\rangle = \langle x,\, \lambda\rangle$, is an isometric embedding, so E is a closed subspace of E''.

Banach algebras

Let A be an algebra such that $(A, \|\cdot\|)$ is also a Banach space. Then A is a **Banach algebra** if also $\|ab\| \leq \|a\| \, \|b\|$ for all $a, b \in A$.

In this case, maximal left ideals are closed, so that rad$A$ is a closed ideal in A.

Example Let S be a non-empty set. Consider the linear space of functions $f : S \to \mathbb{C}$ such that $\sum_{s \in S} |f(s)| < \infty$. This is the space $\ell^1(S)$. It is a Banach space for the norm

$$\|f\|_1 = \sum_{s \in S} |f(s)| \; .$$

Now suppose that S is a semigroup. Then $\ell^1(S)$ is a Banach algebra, where the product is again specified by $\delta_s \star \delta_t = \delta_{st}$ for all $s, t \in S$.

Ideals and semi-simplicity

Triviality Let S be a semigroup, and take a left ideal I in S. Set $J = \overline{\text{lin}}\{\delta_s : s \in I\}$. Then J is a closed left ideal in $\ell^1(S)$. $\square$

Theorem Let S be a group or the semigroup $(\mathbb{N}, +)$. Then $\mathbb{C}S$ and $\ell^1(S)$ are semi-simple algebras. $\square$

There are trivial 2-dimensional examples of semigroups S such that $\mathbb{C}S$ is not semi-simple.

A general classification of semigroups S such that $\mathbb{C}S$ or $\ell^1(S)$ are semi-simple seems to be inaccessible.

Open: Is $\ell^1(\beta\mathbb{N}, \Box)$ semi-simple? Does semi-simplicity of one of $\mathbb{C}S$ and $\ell^1(S)$ imply the same for the other?

Partial results in [DSZZ].

Maximal left ideals

Example Let S be a semi-group. Set

$$\ell_0^1(S) = \left\{ f \in \ell^1(S) : \sum_{s \in S} f(s) = 0 \right\} .$$

This is the **augmentation ideal**. It is a maximal ideal and a maximal left ideal. It may be the only maximal left ideal. □

Exercise Describe the maximal left ideals in $\ell^1(\mathbb{F}_2)$. How many have finite codimension?

A left ideal I in a unital algebra A is **finitely-generated** if there exist $a_1, \dots, a_n \in A$ such that $I = Aa_1 + \cdots + Aa_n$.

Conjecture Let S be a semi-group. Suppose that all maximal left ideals in $\ell^1(S)$ are finitely-generated. Then S is finite.

Proposition (Jared White) $\ell_0^1(S)$ is finitely-generated if and only if S is 'pseudo-finite'. For groups, pseudo-finite = finite. □

$M(\beta S)$

Example Start with a non-empty set S and $E = \ell^1(S)$. Then we can identify E' with $\ell^\infty(S)$, the Banach space of bounded sequences on S. Of course $\ell^\infty(S)$ is identified with $C(\beta S)$. The bidual E'' is $C(\beta S)' = M(\beta S)$, the Banach space of all complex-valued, regular Borel measures μ on βS, with

$$\|\mu\| = |\mu|(\beta S).$$

Clearly $\ell^1(\beta S) \subset M(\beta S)$.

A measure μ is **continuous** if $\mu(\{u\}) = 0$ for all $u \in \beta S$. These measures form a closed linear subspace $M_c(\beta S)$ of $M(\beta S)$, and

$$M(\beta S) = \ell^1(\beta S) \oplus M_c(\beta S).$$

Also $M(\beta S) = \ell^1(S) \oplus M(S^*)$.

Claim Properties of $M(\beta S)$ give information about S.

Biduals of Banach algebras

Let A be a Banach algebra. Then there are two natural products, $\Box$ and $\Diamond$, on the bidual A'' of A; they are called the **Arens products**.

For $\lambda \in A'$ and $a \in A$, define $a \cdot \lambda, \lambda \cdot a \in A'$ by

$$\langle b,\, a\cdot\lambda\rangle = \langle ba,\, \lambda\rangle\,, \quad \langle b,\, \lambda\cdot a\rangle = \langle ab,\, \lambda\rangle \quad (b \in A)\,.$$

[This makes A' into a **Banach** A**-bimodule**.]

For $\lambda \in A'$ and $\Phi \in A''$, define $\lambda \cdot \Phi \in A$ and $\Phi \cdot \lambda \in A'$ by

$$\langle a,\, \lambda \cdot \Phi\rangle = \langle \Phi,\, a \cdot \lambda\rangle\,, \quad \langle a,\, \Phi \cdot \lambda\rangle = \langle \Phi,\, \lambda \cdot a\rangle$$

for $a \in A$. For $\Phi, \Psi \in A''$, define

$$\langle \Phi \Box \Psi,\, \lambda\rangle = \langle \Phi,\, \Psi \cdot \lambda\rangle\,,$$

for $\lambda \in A'$, and similarly for $\Diamond$.

Basic facts

Fact 1 $(A'', \Box)$ and $(A'', \Diamond)$ are Banach algebras containing A as a closed subalgebra.

Fact 2 Let $\Phi, \Psi \in A''$. Then there are nets (a_α) and (b_β) in A with $a_\alpha \to \Phi$ and $b_\beta \to \Psi$ weak-$*$ in A'', and then

$$\Phi \,\Box\, \Psi = \lim_\alpha \lim_\beta a_\alpha b_\beta$$

and also $\Phi \Diamond \Psi = \lim_\beta \lim_\alpha a_\alpha b_\beta$.

The algebra A is **Arens regular** if $\Box$ and $\Diamond$ coincide on A''. All C^*-algebras are Arens regular, but infinite-dimensional group algebras are not.

Biduals of semi-group algebras

Start with a semigroup S and the semigroup algebra $A = (\ell^1(S), \star)$.

Then $A' = \ell^\infty(S) = C(\beta S)$ and $A'' = M(\beta S)$.

We can transfer the Arens products $\Box$ and $\Diamond$ to $M(\beta S)$, and so we can define

$$\mu \Box \nu \text{ and } \mu \Diamond \nu \quad \text{for} \quad \mu, \nu \in M(\beta S)\,.$$

In particular, we define $\delta_u \Box \delta_v$ for $u, v \in \beta S$, and, of course, $\delta_u \Box \delta_v = \delta_{u \Box v}$.

Obviously we can regard βS as a subset of $M(\beta S)$, and then $(\beta S, \Box)$ is a sub-semigroup of the multiplicative semigroup of $(M(\beta S), \Box)$.

It is very rare to have $\mu \Box \nu = \nu \Box \mu$.

For example, there are just two points a and b in $\mathbb{N}^*$ such that the only elements ν in $M(\beta\mathbb{N})$ with both $\delta_a \Box \nu = \nu \Box \delta_a$ and $\delta_b \Box \nu = \nu \Box \delta_b$ are already in $\ell^1(\mathbb{N})$. See [DLS].

Left-invariant means

We know that $K = K(\beta\mathbb{N}, \Box)$ is big. How to characterize it?

Let S be a semigroup, and take $\mu \in M(\beta S)$. Then μ is a **mean** if

$$\|\mu\| = \langle 1, \mu \rangle = 1 \,,$$

and μ is **left-invariant** if $s \,\Box\, \mu = \mu \ \ (s \in S)$. The semigroup S is **left-amenable** if there is a left-invariant mean on S, and **amenable** if there is a mean that is left and right invariant.

The sets of means and of left-invariant means on S are denoted by $\mathfrak{M}(S)$ and $\mathfrak{L}(S)$.

Both are weak-$*$-compact, convex subsets of $(M(\beta S), \Box)$. Further, $\mathfrak{M}(S)$ is a sub-semigroup, and hence is a compact, right topological semigroup in $(M(\beta S), \Box)$, so it has a minimum ideal $K(\mathfrak{M}(S))$.

Left-amenable semigroups

A left-amenable group is amenable. All abelian semigroups are amenable; $\mathbb{F}_2$ is not amenable; it is a very famous open question whether Thompson's group is amenable.

A left or right ideal in a left-amenable semigroup is itself left amenable.

Let G be an amenable group of cardinality κ. Then $|\mathfrak{L}(G)| = 2^{2^\kappa}$, but there are semigroups S with $|\mathfrak{L}(S)| = 1$.

Fact Suppose that S is left-amenable. Then $\mathfrak{L}(S) = K(\mathfrak{M}(S), \Box)$. $\Box$

Question Characterize $K(\mathfrak{M}(S), \Box)$ when S is **not** left-amenable. Relate $K(\mathfrak{M}(S), \Box)$ and $K(\mathfrak{M}(S), \Diamond)$, especially when $S = \mathbb{F}_2$. $\Box$

The support of measures

Let S be a semigroup, and take $\mu \in M(\beta S)$. Then μ has a **support**, $\operatorname{supp}\mu$. Suppose that $\mu \in \mathfrak{M}(S)$. Then it can be shown that

$$\operatorname{supp}\mu = \bigcap\{\overline{F} : F \subset S, \langle \mu, \chi_F \rangle = 1\}\,.$$

Suppose that S is left-amenable. Then $\operatorname{supp}\mu$ is a closed left ideal in βS for each $\mu \in \mathfrak{L}(S)$. We define

$$L(\beta S) = \bigcup\{\operatorname{supp}\mu : \mu \in \mathfrak{L}(S)\}\,,$$

a left ideal in βS. Is it closed?

Suppose that (μ_n) is a sequence in $\mathfrak{L}(S)$, and set $\mu = \sum_{n=1}^{\infty} \mu_n/2^n$. Then

$$\operatorname{supp}\mu = \overline{\bigcup\{\operatorname{supp}\mu_{\mathrm{n}} : \mathrm{n} \in \mathbb{N}\}}\,,$$

so $\mathfrak{L}(S)$ contains the closure of any countable subset, but we do not know whether $\mathfrak{L}(S)$ is always closed.

For S infinite and cancellative, $\overline{L(\beta S)} \subset S^*$.

Some ideals in $(\beta S, \Box)$

Fact Let S be a semigroup. Then $\overline{K(\beta S)}$ is an ideal in $(\beta S, \Box)$. [HS]. $\Box$

Theorem Let S be a left-amenable semigroup. Then:

(i) $L(\beta S)$ and $\overline{L(\beta S)}$ are ideals in $(\beta S, \Box)$;

(ii) $K(\beta S) \subset L(\beta S)$, and so $\overline{K(\beta S)} \subset \overline{L(\beta S)}$. $\Box$

More ideals

Let S be a cancellative semigroup (so S^* is a semigroup).

Definition Set

$$S^*_{[n]} = \{u_1 \Box \cdots \Box u_n : u_1, \dots, u_n \in S^*\}\,.$$

Thus $(S^*_{[n]})$ is a decreasing nest of ideals in S^*, and

$$E(S^*) \cup K(\beta S) \subset S^*_{[\infty]} := \bigcap S^*_{[n]}\,.$$

Also $(\overline{S^*_{[n]}})$ is a decreasing nest of closed ideals.

Fact Each $\overline{S^*_{[n]}}$ is a closed ideal, and $\overline{S^*_{[2]}} \neq S^*$. [DLS] $\Box$

Definition Set $T^*_{[1]} = S^*$ and $T^*_{[n+1]} = \overline{S^* \Box T^*_{[n]}}$ for $n \in \mathbb{N}$, so that $\overline{S^*_{[n]}} \subset T^*_{[n]}$.

The latter look the same; $\overline{S^*_{[2]}} = T^*_{[2]}$. But it is not clear whether $\overline{S^*_{[3]}} = T^*_{[3]}$ - see later.

Relations with $L(\beta S)$

Theorem [DLS] Let S be infinite, left-amenable, and cancellative (e.g., $S = \mathbb{N}$). Then $\overline{L(\beta S)} \subset \overline{S^*_{[\infty]}}$. $\Box$

Proposition [HS] The set $\overline{E(K(\beta\mathbb{N}))}\setminus\mathbb{N}^*_{[2]}$ is infinite, and so $K(\beta\mathbb{N})$ and $\mathbb{N}^*_{[2]}$ are not closed. $\square$

Proposition [DLS] (i) It is not true that $L(\beta\mathbb{N}) \subset \mathbb{N}^*_{[2]}$.

(ii)There are idempotents in $\mathbb{N}^*$ that are not in $\overline{L(\beta\mathbb{N})}$, and so $\overline{L(\beta\mathbb{N})} \subsetneq \overline{\mathbb{N}^*_{[\infty]}}$. $\square$

Question Is $L(\beta\mathbb{N})$ closed in $\beta\mathbb{N}$?

Proposition [DLS] $\overline{K(\beta\mathbb{N})} \subsetneq \overline{L(\beta\mathbb{N})}$ (and so $K(\beta\mathbb{N}) \subsetneq L(\beta\mathbb{N})$). $\square$

The above use the following.

For a subset U of $\mathbb{N}$, the **upper density** of U is

$$\overline{d}(U) = \limsup_{n\to\infty} |U \cap \mathbb{N}_n| / n \,.$$

Now regard $u \in \beta\mathbb{N}$ as an ultrafilter on $\mathbb{N}$, and set

$$\Delta = \{u \in \beta\mathbb{N} : \overline{d}(U) > 0 \ \ (u \in U)\}.$$

Then Δ is a closed left ideal in $(\beta\mathbb{N}, \Box\,)$.

A theorem of Hindman

Neil Hindman showed us the following surprising fact (and more); see [DLS].

Theorem $\overline{\mathbb{N}^*_{[k+1]}} \subsetneq \overline{\mathbb{N}^* \Box \overline{\mathbb{N}^*_{[k]}}}$ for all $k \geq 2$.

Starting point Each $n \in \mathbb{N}$ has a unique expression in the form

$$n = \sum_{i=1}^{\infty} \varepsilon_i(n) 2^i\,,$$

with $\varepsilon_i(n) \in \{0, 1\}$ and $= 0$ eventually. Then some combinatorics. $\Box$

Ideals in $M(\beta S, \Box)$

Let S be a semigroup.

Fact Let L be a closed left ideal in $(\beta S, \Box)$. Then $M(L)$ is a weak-$*$-closed left ideal in $(M(\beta S), \Box)$. $\Box$

Fact However $M(R)$ is not necessarily a right ideal in $(M(\beta S), \Box)$ whenever R is a closed right ideal in βS. Indeed, the closed subspace $M(\overline{K(\beta \mathbb{N})})$ of $M(\beta \mathbb{N})$ is a left ideal, but not a right ideal, in $(M(\beta \mathbb{N}), \Box)$. $\Box$

Fact Let S be cancellative. Then each of $M(\overline{S^*_{[n]}})$ is a weak-$*$ closed left ideal. Further, each of $M(T^*_{[n]})$ is a weak-$*$-closed (two-sided) ideal. [DLS] $\Box$

Question Recall that maybe $\overline{S^*_{[3]}} \subsetneq T^*_{[3]}$. Is $M(\overline{S^*_{[3]}})$ always a right ideal in $(M(\beta S), \Box)$? In particular, what if $S = \mathbb{N}$?