Crystal monoids

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Plactic monoid: Three sides of the same coin

Let $\mathcal{A}_n$ be the finite ordered alphabet $\{1 < 2 < \ldots < n\}$.

I want to give three different ways of defining a certain congruence $\sim$ on the free monoid $\mathcal{A}_n^*$:

1. Presentation (Knuth relations)
2. Tableaux (Schensted insertion algorithm)
3. Crystal bases (in the sense of Kashiwara)

We call $\sim$ the Plactic congruence and the resulting quotient monoid $\mathrm{Pl}(A_n) = \mathcal{A}_n^* / \sim$ is called the Plactic monoid (of rank n).

The Plactic monoid

- Has origins in work of Schensted (1961) and Knuth (1970) concerned with combinatorial problems on Young tableaux.
- Later studied in depth by Lascoux and Shützenberger (1981).

Due to close relations to Young tableaux, has become a tool in several aspects of representation theory and algebraic combinatorics.

Applications of the Plactic monoid

- proof of the Littlewood–Richardson rule for Schur functions (an important result in the theory of symmetric functions);
  - see appendix of J. A. Green's "Polynomial representations of GL_n".
- a combinatorial description of the Kostka–Foulkes polynomials, which arise as entries of the character table of the finite linear groups.

M. P. Schützenberger 'Pour le monoïde plaxique' (1997)

Argues that the Plactic monoid ought to be considered as "one of the most fundamental monoids in algebra".

Plactic monoid via Knuth relations

Definition

Let $\mathcal{A}_n$ be the finite ordered alphabet $\{1 < 2 < \ldots < n\}$.
Let $\mathcal{R}$ be the set of defining relations:

$$\begin{aligned} zxy &= xzy \quad \text{and} \quad yzx = yxz && x < y < z, \\ xyx &= xxy \quad \text{and} \quad xyy = yxy && x < y. \end{aligned}$$

The Plactic monoid $\text{Pl}(A_n)$ is the monoid defined by the presentation $\langle \mathcal{A}_n | \mathcal{R} \rangle$.

That is, $\text{Pl}(A_n) = \mathcal{A}_n^* / \sim$ where $\sim$ is the smallest congruence on the free monoid $\mathcal{A}_n^*$ containing $\mathcal{R}$.

- This is the most efficient way to define the Plactic congruence $\sim$.
- The relations in this presentation are called the Knuth relations.

A (semi-standard) tableau

1	1	1	2	2	4	4
2	2	3	3			
4	5	5	6			
6	8					

Properties

- Is a filling of the Young diagram with symbols from the alphabet $\mathcal{A}_n$.
- Rows read left-to-right are non-decreasing.
- Columns read down are strictly increasing.
- Never have a longer row below a strictly shorter one.

Schensted column insertion algorithm

- Associates to each word $w \in \mathcal{A}_n^*$ a tableau $P(w)$.
- The algorithm which produces $P(w)$ is recursive.

Input: Any letter $x \in \mathcal{A}_n$ and a tableau T.

Output: A new tableau denoted $x \to T$.

The idea: Suppose $T = C_1 C_2 \ldots C_r$ where the C_i are the columns of T.

- We try to insert the box $\boxed{x}$ under the column C_1 if we can.
- If this fails, the box $\boxed{x}$ will be put into column C_1 higher up (in an appropriate place) and will "bump out" a box $\boxed{y}$.
- We then take the bumped out box $\boxed{y}$ and try and insert it under the column C_2, and so on...

Schensted's column insertion algorithm

Algorithm:

- If $T = \varnothing$ then $x \to T = \boxed{x}$
- If $T = C$ has only on column then

$$x \to T = \begin{cases} \begin{array}{|c|}\hline C \\ \hline x \\ \hline \end{array} & \text{if } \begin{array}{|c|}\hline C \\ \hline x \\ \hline \end{array} \text{ is a column} \\ \\ \begin{array}{|c|c|}\hline C' & y \\ \hline \end{array} & \text{otherwise} \end{cases}$$

  where y is the minimal letter in C such that $x \leq y$ and $C' = C - \{y\} + \{x\}$.

- If $T = C_1 C_2 \ldots C_r$ has $r \geq 2$ columns

$$x \to T = \begin{cases} \begin{array}{|c|}\hline C_1 \\ \hline x \\ \hline \end{array}\, C_2 \ldots C_r & \text{if } x \to C_1 = \begin{array}{|c|}\hline C_1 \\ \hline x \\ \hline \end{array} \\ \\ C_1'(y \to C_2 \ldots C_r) & \text{if } x \to C_1 = \begin{array}{|c|c|}\hline C_1' & y \\ \hline \end{array} \end{cases}$$

Schensted's column insertion algorithm

Example

$\mathcal{A}_4 = \{1 < 2 < 3 < 4\}$ if $w = 232143$ then $P(w)$ is obtained as:

$$\begin{array}{|c|}\hline 2\\\hline\end{array}\,,\ \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\,,\ \begin{array}{|c|c|}\hline 2 & 2\\\hline 3 \\\cline{1-1}\end{array}\,,\ \begin{array}{|c|c|c|}\hline 1 & 2 & 2\\\hline 3 \\\cline{1-1}\end{array}\,,\ \begin{array}{|c|c|c|}\hline 1 & 2 & 2\\\hline 3 \\\cline{1-1} 4 \\\cline{1-1}\end{array}\,,\ \begin{array}{|c|c|c|}\hline 1 & 2 & 2\\\hline 3 & 3 \\\cline{1-2} 4 \\\cline{1-1}\end{array} = P(w).$$

Observation: $231 = 213$ is one of the Knuth relations in the presentation of the Plactic monoid and $P(231) = P(213)$:

$$\begin{array}{|c|}\hline 2\\\hline\end{array}\,,\ \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\,,\ \begin{array}{|c|c|}\hline 1 & 2\\\hline 3 \\\cline{1-1}\end{array} = P(231), \qquad \begin{array}{|c|}\hline 2\\\hline\end{array}\,,\ \begin{array}{|c|c|}\hline 1 & 2\\\hline\end{array}\,,\ \begin{array}{|c|c|}\hline 1 & 2\\\hline 3 \\\cline{1-1}\end{array} = P(213).$$

Theorem (Lascoux and Shützenberger (1981))

Define a relation $\sim$ on $\mathcal{A}_n^*$ by

$$u \sim w \Leftrightarrow P(u) = P(w).$$

Then $\sim$ is the Plactic congruence and $\mathrm{Pl}(A_n) = \mathcal{A}_n^* / \sim$ is the Plactic monoid.

The Plactic monoid via tableaux

$\mathrm{w}(T)$ = the word obtained by reading the columns of a tableau T from right to left and top to bottom (Japanese reading).

Example: If $T = \begin{array}{|c|c|c|}\hline 1 & 1 & 4 \\ \hline 2 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$ then $\mathrm{w}(T) = 415123$.

Theorem (Lascoux and Shützenberger (1981))

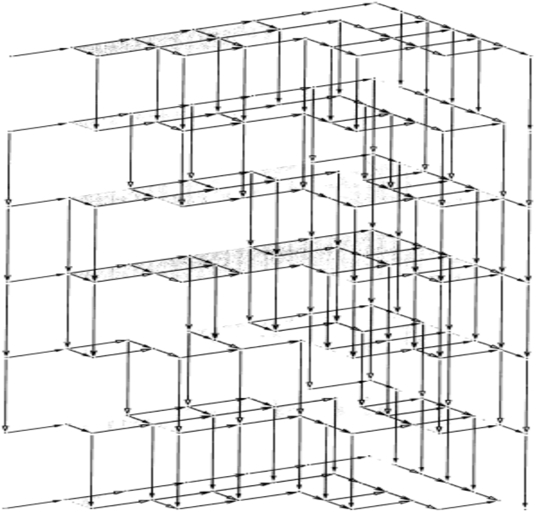
For any word $u \in \mathcal{A}_n^*$ we have

- $u = \mathrm{w}(P(u))$ in the Plactic monoid $\mathrm{Pl}(A_n)$ and
- $P(u)$ is the unique tableau such that this is true.

Conclusion: the set of word readings of tableaux is a set of normal forms for the elements of the Plactic monoid. So, the Plactic monoid is the monoid of tableaux:

Elements	The set of all tableaux over $\mathcal{A}_n = \{1 < 2 < \cdots < n\}$.
Products	Computed using Schensted insertion.

Crystals

 [1]

[1] Fig 8.4 from Hong and Kang's book *An introduction to quantum groups and crystal bases.*

Crystal graphs

(following Kashiwara and Nakashima (1994))

Idea: Define a directed labelled digraph Γ_{A_n} with the properties:

- Vertex set = $\mathcal{A}_n^*$
- Each directed edge is labelled by a symbol from the label set $I = \{1, 2, \ldots, n-1\}$.
- For each vertex $u \in \mathcal{A}_n^*$ every $i \in I$ there is at most one directed edge labelled by i leaving u, and there is at most one directed edge labelled by i entering u,

$$u \xrightarrow{\quad i \quad} v \; , \qquad w \xrightarrow{\quad i \quad} u$$

- If $u \xrightarrow{\quad i \quad} v$ then $|u| = |v|$, so words in the same component have the same length as each other. In particular, connected components are all finite.

Building the crystal graph Γ_{A_n}

$$\mathcal{A}_n = \{1 < 2 < \ldots < n\}$$

We begin by specifying structure on the words of length one

$$1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \xrightarrow{n-2} n-1 \xrightarrow{n-1} n$$

This is known as a Crystal basis.

Kashiwara operators

For each $i \in \{1, \ldots, n-1\}$ we define partial maps $\tilde{e}_i$ and $\tilde{f}_i$ on the letters $\mathcal{A}_n$ called the Kashiwara crystal graph operators. For each edge

$$a \xrightarrow{i} b \ ,$$

we define $\tilde{f}_i(a) = b$ and $\tilde{e}_i(b) = a$.

The crystal graph Γ_{A_n}

$$1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \xrightarrow{n-2} n-1 \xrightarrow{n-1} n$$

$$a \xrightarrow{i} \tilde{f}_i(a), \qquad \tilde{e}_i(b) \xrightarrow{i} b$$

Kashiwara operators on words

For $u, v \in \mathcal{A}_n^+$ define inductively

$$\tilde{e}_i(uv) = \begin{cases} u\,\tilde{e}_i(v) & \text{if } \varphi_i(u) < \epsilon_i(v) \\ \tilde{e}_i(u)\,v & \text{if } \varphi_i(u) \geq \epsilon_i(v) \end{cases}, \quad \tilde{f}_i(uv) = \begin{cases} \tilde{f}_i(u)\,v & \text{if } \varphi_i(u) > \epsilon_i(v) \\ u\tilde{f}_i(v) & \text{if } \varphi_i(u) \leq \epsilon_i(v) \end{cases}.$$

where ϵ_i and φ_i are auxiliary maps defined by

$$\epsilon_i(w) = \max\{k \in \mathbb{N} \cup \{0\} : \underbrace{\tilde{e}_i \cdots \tilde{e}_i}_{k \text{ times}}(w) \text{ is defined}\}$$

$$\varphi_i(w) = \max\{k \in \mathbb{N} \cup \{0\} : \underbrace{\tilde{f}_i \cdots \tilde{f}_i}_{k \text{ times}}(w) \text{ is defined}\}$$

The crystal graph Γ_{A_n}

Definition

The crystal graph Γ_{A_n} is the directed labelled graph with:

- Vertex set: $\mathcal{A}_n^*$
- Directed labelled edges: for $u \in \mathcal{A}_n^*$

$$u \xrightarrow{i} \tilde{f}_i(u), \qquad \tilde{e}_i(u) \xrightarrow{i} u$$

Note: When defined $\tilde{e}_i(\tilde{f}_i(u)) = u$ and $\tilde{f}_i(\tilde{e}_i(u)) = u$.

Practical computation of $\tilde{e}_i(u)$ and $\tilde{f}_i(u)$

Let $u \in \mathcal{A}_n^*$ and $i \in I$.

Question: Are either / both of the following edges in Γ_{A_n}?

$$u \xrightarrow{i} \tilde{f}_i(u), \qquad \tilde{e}_i(u) \xrightarrow{i} u$$

Algorithm:

- Under each letter a of w write:
  - $\epsilon_i(a)$ times the symbol $-$ and $\varphi_i(a)$ times the symbol $+$.
- Take the resulting string of $-$'s and $+$'s and delete all adjacent $+-$.
- The resulting string is then $-^{\epsilon_i(w)}+^{\varphi_i(w)}$.
- $\tilde{e}_i(w)$: obtained by applying $\tilde{e}_i$ to the letter a above the rightmost remaining $-$, if it exists.
- $\tilde{f}_i(w)$: obtained by applying $\tilde{f}_i$ to the letter a above the leftmost remaining $+$, if it exists.

Example with $\mathcal{A}_3 = \{1 < 2 < 3\}$

$$1 \xrightarrow{\;1\;} 2 \xrightarrow{\;2\;} 3$$

$$a \xrightarrow{\;i\;} \tilde{f}_i(a) \quad , \qquad \tilde{e}_i(b) \xrightarrow{\;i\;} b$$

Example

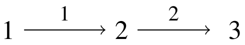
Let $u = 33212313232$ and let $i = 2 \in I = \{1, 2\}$.

$$\begin{array}{ccccccccccc}
3 & 3 & 2 & 1 & 2 & 3 & 1 & 3 & 2 & 3 & 2 \\
- & - & + & & + & - & & - & + & - & + \\
- & - & \cancel{+} & & \cancel{+} & \cancel{-} & & \cancel{-} & \cancel{+} & \cancel{-} & + \\
- & {\color{blue}-} & & & & & & & & & {\color{blue}+}
\end{array}$$

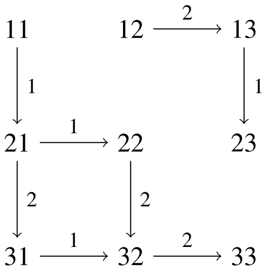
$$\begin{array}{ccccccccccl}
3 & 3 & 2 & 1 & 2 & 3 & 1 & 3 & 2 & 3 & {\color{blue}3} = \tilde{f}_2(u) \\
3 & {\color{blue}2} & 2 & 1 & 2 & 3 & 1 & 3 & 2 & 3 & 2 = \tilde{e}_2(u)
\end{array}$$

Crystal graph components for $\mathcal{A}_3 = \{1 < 2 < 3\}$

Word length one

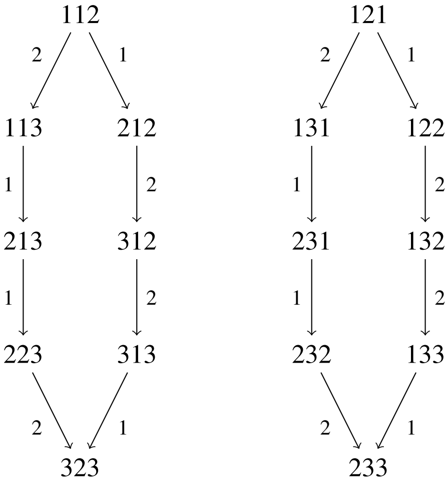


Word length two



Crystal graph components for $\mathcal{A}_3 = \{1 < 2 < 3\}$

Word length three



Plactic monoid via crystals

Definition: Two connected components $B(w)$ and $B(w')$ of Γ_{A_n} are isomorphic if there is a label-preserving digraph isomorphism $f : B(w) \to B(w')$.

Fact: In Γ_{A_n} if $B(w) \cong B(w')$ then there is a unique isomorphism $f : B(w) \to B(w')$.

Theorem (Kashiwara and Nakashima (1994))

Let Γ_{A_n} be the crystal graph with crystal basis

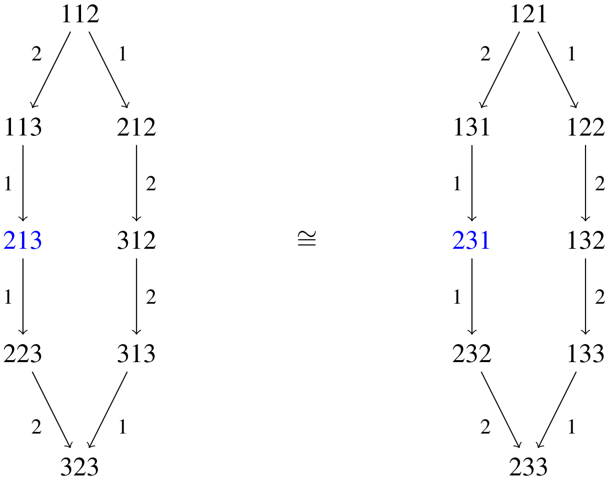
$$1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \xrightarrow{n-2} n-1 \xrightarrow{n-1} n$$

Define a relation $\sim$ on $\mathcal{A}_n^*$ by

$$u \sim w \Leftrightarrow \exists \text{ an isomorphism } f : B(u) \to B(w) \text{ with } f(u) = w.$$

Then $\sim$ is the Plactic congruence and $\mathrm{Pl}(A_n) = \mathcal{A}_n^* / \sim$ is the Plactic monoid.

Crystal graph components for $\mathcal{A}_3 = \{1 < 2 < 3\}$



(**Confession:** I lied a bit. Actually, crystal isomorphisms must also preserve "weight". For $\mathrm{Pl}(A_n)$ weight preserving means "content preserving".)

Where do crystals come from?

WARNING!

Lie algebras are not algebras

Quantum groups are not groups

and

Good enough is not good enough

Where do crystals come from?

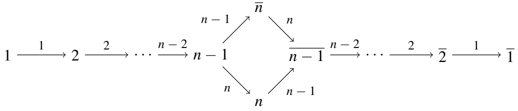
J. Hong, S.-J. Kang,
Introduction to Quantum Groups and Crystal Bases.
Stud. Math., vol. 42, Amer. Math. Soc., Providence, RI, 2002.

- Take a “nice” Lie algebra $\mathfrak{g}$. Nice means symmetrizable Kac-Moody Lie algebra e.g. a finite-dimensional semisimple Lie algebra.
- From $\mathfrak{g}$ construct its universal enveloping algebra $U(\mathfrak{g})$ which is an associative algebra.
- Drinfeld and Jimbo (1985): defined q-analogues $U_q(\mathfrak{g})$, quantum deformations, with parameter q
  - $q = 1$: $U_q(\mathfrak{g})$ coincides with $U(\mathfrak{g})$
  - $q = 0$: is called crystallisation (Kashiwara (1990)). The parameter q corresponds to temperature, $q = 0$ is absolute temperature zero.

Where do crystals come from?

- Crystal bases are bases of $U_q(\mathfrak{g})$-modules at $q = 0$ that satisfy certain axioms.
  - Kashiwara (1991): proves existence and uniqueness of crystal bases of finite dimensional representations of $U_q(\mathfrak{g})$.
- Every crystal basis has the structure of a coloured digraph (called a crystal graph). The structure of these coloured digraphs has been explicitly determined for certain semisimple Lie algebras (special linear, special orthogonal, symplectic, some exceptional types).
- The crystal constructed from the crystal basis using Kashiwara operators is then a useful combinatorial tool for studying representations of $U_q(\mathfrak{g})$.
  - e.g. Gives information about decomposing tensor products of finite dimensional $U_q(\mathfrak{g})$-modules into direct sums of irreducible components.

Crystal bases and crystal monoids

Lie algebra type	Crystal basis	Monoid
A_n: $\mathfrak{sl}_{n+1}$	$1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \xrightarrow{n-2} n-1 \xrightarrow{n-1} n$	$\mathrm{Pl}(A_n)$
B_n: $\mathfrak{so}_{2n+1}$	$1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \xrightarrow{n-1} n \xrightarrow{n} 0 \xrightarrow{n} \overline{n} \xrightarrow{n-1} \cdots \xrightarrow{2} \overline{2} \xrightarrow{1} \overline{1}$	$\mathrm{Pl}(B_n)$
C_n: $\mathfrak{sp}_{2n}$	$1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \xrightarrow{n-1} n \xrightarrow{n} \overline{n} \xrightarrow{n-1} \cdots \xrightarrow{2} \overline{2} \xrightarrow{1} \overline{1}$	$\mathrm{Pl}(C_n)$
D_n: $\mathfrak{so}_{2n}$		$\mathrm{Pl}(D_n)$
G_2	$1 \xrightarrow{1} 2 \xrightarrow{2} 3 \xrightarrow{1} 0 \xrightarrow{1} \overline{3} \xrightarrow{2} \overline{2} \xrightarrow{1} \overline{1}$	$\mathrm{Pl}(G_2)$

Crystal monoids in general

Combinatorial crystals

- Crystal basis = finite labelled directed graph, vertex set X, label set I, satisfying certain axioms so that Kashiwara operators $\tilde{e}_i$, $\tilde{f}_i$ $(i \in I)$ and functions ϵ_i and φ_i make sense.
- A weight function $\mathrm{wt} : X \to P$ where P is some finitely generated free abelian group.
- Construct a (weighted) crystal graph Γ_X from this data
  - Vertex set: X^*
  - Directed labelled edges: determined by $\tilde{e}_i, \tilde{f}_i$

Definition (Crystal monoid)

Let Γ_X be a crystal graph. Define $\approx$ on X^* where $u \approx v$ if there is a (weight preserving) isomorphism $\theta : B(u) \to B(v)$ with $\theta(u) = v$. Then $\approx$ is a congruence on X^* and $X^*/\approx$ is called the crystal monoid of Γ_X.

Known results and our interest

Known results on crystals A_n, B_n, C_n, D_n, or G_2 and their crystal monoids:

1. Crystal bases - combinatorial description Kashiwara and Nakashima (1994).
2. Tableaux theory and Schensted-type insertion algorithms - Kashiwara and Nakashima (1994), Lecouvey (2002, 2003, 2007).
3. Finite presentations for $\mathrm{Pl}(X)$ via Knuth-type relations - Lecouvey (2002, 2003, 2007).

Theory we have been developing for crystal monoids:

4. Finite complete rewriting systems
5. Automatic structures

A. J. Cain, R. D. Gray, A. Malheiro
Crystal bases, finite complete rewriting systems, and biautomatic structures for Plactic monoids of types A_n, B_n, C_n, D_n, and G_2.
arXiv:math.GR/1412.7040, 50 pages.

Complete rewriting systems

X - alphabet, $R \subseteq X^* \times X^*$ - rewrite rules, $\langle X \mid R \rangle$ - rewriting system
Write $r = (r_{+1}, r_{-1}) \in R$ as $r_{+1} \to r_{-1}$.

Define a binary relation $\to_R$ on X^* by

$$u \to_R v \;\Leftrightarrow\; u \equiv w_1 r_{+1} w_2 \text{ and } v \equiv w_1 r_{-1} w_2$$

for some $(r_{+1}, r_{-1}) \in R$ and $w_1, w_2 \in X^*$.

$\xrightarrow{*}_R$ is the transitive and reflexive closure of $\to_R$

Noetherian: No infinite descending chain

$$w_1 \to_R w_2 \to_R \cdots \to_R w_n \to_R \cdots$$

Confluent: Whenever

$$u \xrightarrow{*}_R v \text{ and } u \xrightarrow{*}_R v'$$

there is a word $w \in X^*$:

$$v \xrightarrow{*}_R w \text{ and } v' \xrightarrow{*}_R w$$

Definition: $\langle X \mid R \rangle$ is a finite complete rewriting system if it is complete (noetherian and confluent) and $|X| < \infty$ and $|R| < \infty$.

Finite complete rewriting systems

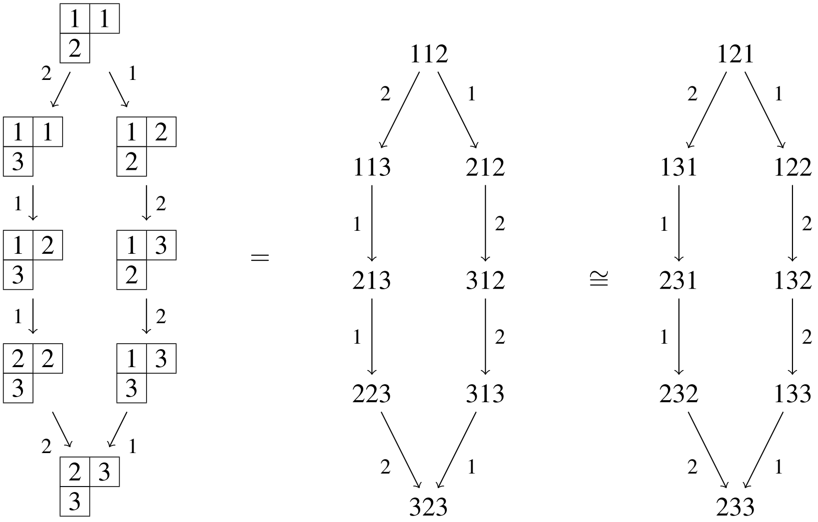
Theorem (Cain, RG, Malheiro (2014))

For any $X \in \{A_n, B_n, C_n, D_n, G_2\}$, there is a finite complete rewriting system (Σ, T) that presents $\mathrm{Pl}(X)$.

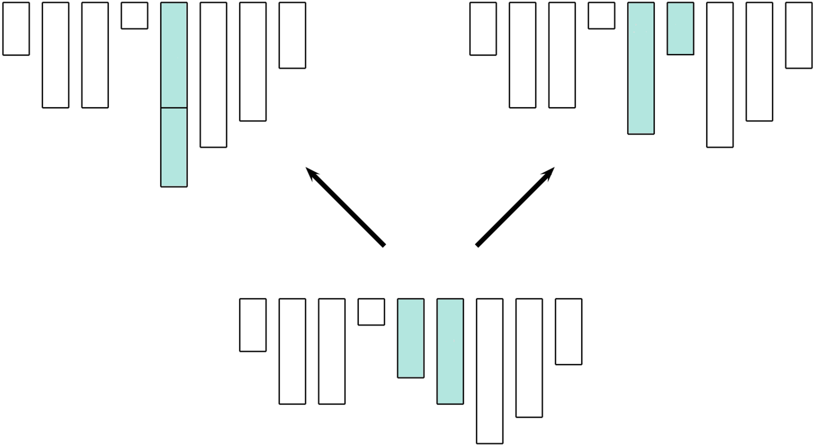
Notes on the proof:

- Builds on our earlier results on the Plactic monoid $\mathrm{Pl}(A_n)$.
- In each case there is a tableau theory. Admissible columns are columns of tableaux.
- *Key idea:* work with the larger generating set of admissible columns.
- A tabloid is a sequence of admissible columns.
- The rewriting system takes a tabloid and rewrites it by multiplying adjacent pairs of admissible columns.
- Kashiwara operators preserve shapes of tabloids so it suffices to consider pairs of columns whose readings are highest-weight words.

Crystal graph components and tableaux

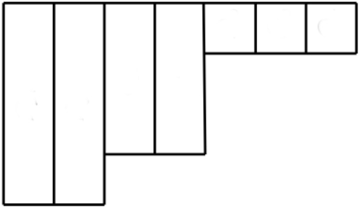


Rewriting tabloids



- Multiplying two adjacent admissible columns of a tabloid brings us one step closer to being a tableau.

Complete rewriting system



- Rewriting converges to the unique tableau representative of the element.

Automatic structures

Automatic groups and monoids

- Automatic groups
  - Capture a large class of groups with easily solvable word problem
  - Examples: finite groups, free groups, free abelian groups, various small cancellation groups, Artin groups of finite and large type, Braid groups, hyperbolic groups.
- Automatic semigroups and monoids
  - Classes of monoids that have been shown to be automatic include divisibility monoids and singular Artin monoids of finite type.

Defining property: existence of rational set of normal forms (with respect to some finite generating set A) such that $\forall a \in A$, there is a finite automaton recognising pairs of normal forms that differ by multiplication by a.

Proposition (Campbell et al. (2001))

Automatic monoids have word problem solvable in quadratic time.

Automaticity

Theorem (Cain, RG, Malheiro (2014))

The monoids $\mathrm{Pl}(A_n)$, $\mathrm{Pl}(B_n)$, $\mathrm{Pl}(C_n)$, $\mathrm{Pl}(D_n)$, and $\mathrm{Pl}(G_2)$ are all biautomatic.

- Biautomatic = the strongest form of automaticity for monoids.
- The language of representatives of the biautomatic structure is the language of irreducible words of the rewriting systems (Σ, T) constructed above.
- As above, crystal bases theory can be used to reduce the problem to just considering highest-weight words.

Current and future work

- Further develop the theory of crystal monoids in general
  - We can obtain other examples (e.g. bicyclic monoid is a crystal monoid).
  - They all have decidable word problem.
  - Under what conditions do they admit finite complete rewriting systems / are automatic?
- What do our results say about the Plactic algebras of Littelmann?

P. Littelmann,
A Plactic Algebra for Semisimple Lie Algebras.
Advances in Mathematics 124 (1996), 312–331.

- Investigate how our results might be applied to give new computational tools for working with crystals (e.g. using rewriting systems / finite automata to compute with crystals).