Graph Automatic Semigroups

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Regular languages

An alphabet is a set Σ of symbols.The elements of Σ are letters. A word over our alphabet is a string of letters from Σ, and the set of all possible words over an alphabet is denoted Σ^*. This contains the empty word ϵ. A language is any subset of Σ^*.

Regular languages over Σ can be defined recursively:

- $\emptyset$, $\{\epsilon\}$, and $\{a\}$ for each $a \in \Sigma$ are regular, and
- if A and B are regular languages over Σ then $A \cup B$, $A \cdot B$, and A^* are regular.

Regular languages are precisely those which are accepted by finite state automata.

Graph automatic semigroups

Graph automatic groups were introduced by Kharlampovich, Khoussainov and Miasnikov. Their definition is naturally applicable to semigroups.

Definition

Let S be a semigroup generated by a finite set X. We call S *(Cayley) graph automatic* iff there exisits a finite alphabet Σ, a regular language $R \subseteq \Sigma^*$, and an onto map $\nu : R \to S$ such that the sets

$$R_= = \{(\alpha, \beta) \in R \times R : \nu(\alpha) = \nu(\beta)\}$$

and

$$R_x = \{(\alpha, \beta) \in R \times R : \nu(\alpha)x = \nu(\beta)\}$$

for $x \in X$ are regular.

We say that (X, Σ, R, ν) is a *graph automatic structure* for S.

Automatic semigroups

Automatic semigroups are those where our alphabet is equal to our generating set, and so the map becomes a homomorphism.

Examples of automatic semigroups include the bicyclic monoid and any finitely generated free semigroup.

The Heisenberg group $\mathscr{H}_3(\mathbb{Z})$, consisting of matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

is graph automatic, but not automatic.

FA-presentable structures

A structure is FA-presentable if its underlying set and all its relations (including equality) can be recoginised by finite state automata.

Examples of FA-presentable semigroups include the bicyclic monoid and $(\mathbb{N}, +)$.

Graph automatic semigroups are those whose Cayley graphs have an automatic presentation.

The free semigroup on more than one generator is graph automatic but not FA-presentable.

Examples

All finite semigroups are graph automatic.

Automatic semigroups are graph automatic.

Finitely generated FA-presentable semigroups are graph automatic.

There exist semigroups which are graph automatic, but neither automatic nor FA-presentable, for example $\mathscr{H}_3(\mathbb{Z}) * F_2$.

Properties of graph automatic semigroups

Graph automaticity is independent of generating set.

If a structure is FA-presentable, then it has an injective automatic presentation (Khoussainov and Nerode).
Given a graph automatic structure, we can convert it to a structure with uniqueness by taking just the shortest word representing each element with respect to the shortlex order.

If a structure is FA-presentable, then it has a binary automatic presentation (Blumensath).

So any graph automatic semigroup has an injective binary graph automatic structure.

Graph automatic semigroups have quadratic word problem.

Subsemigroups and ideals

A subsemigroup of a graph automatic semigroup is regular if its preimage is a regular language.

Theorem

Regular subsemigroups of graph automatic semigroups are graph automatic.

Corollary

Finitely generated right ideals of graph automatic semigroups are graph automatic.

Constructions

Let S and T be graph automatic semigroups. Then

- S^1 and S^0 are graph automatic.
- Any subsemigroup of S of finite Rees index is graph automatic.
- $S \times T$ is graph automatic, if finitely generated.
- $S * T$ is graph automatic.
- Any finitely generated Rees matrix semigroup $M[S; I, J; P]$ is graph automatic.

Unary graph automatic semigroups

Definition

A semigroup is *unary graph automatic* if it has a graph automatic structure $(X, \{a\}, R, \nu)$, that is a graph automatic structure over a single letter alphabet.

All finite semigroups are unary graph automatic.

If an infinite structure is unary FA-presentable, then in has an injective unary automatic presentation over the language a^* (Cain, Ruškuc, Thomas). So for infinite semigroups, we need only consider whether we have an injective unary graph automatic structure $(X, a, a^*, \nu : a^* \to S)$.

Unary automata

Consider the format of the acceptor automata for a unary graph automatic semigroup. These will be automata over the alphabet $\{(a, a), (a, \$), (\$, a)\}$.

We assume that our automata are deterministic and we consider only those paths which lead to the accept states.

Note that we must have at least one circuit or loop, as our language is infinite, and if we read a \$ in a component then we must continue to read only \$ afterwards.

Each automaton must accept a^n in the first component precisely once for each $n \in \mathbb{N}$.

Our automata have the following restrictions:

- An acceptor automaton cannot have a $(\$, a)$ circuit.
- An acceptor automaton cannot have two consecutive circuits.
- If an acceptor automaton has two distinct circuits then these circuits must have the same length p. Moreover, each accept state will accept words of a different remainder when their lengths are considered modulo p.
- If an acceptor automaton consists of only one circuit labelled $(a, \$)$ then this circuit has length one, i.e. is a loop.

This means that an acceptor automaton for a unary graph automatic semigroup has one of the following formats:

1. A finite path followed by a single loop of the form $(a, \$)$.
2. A finite path followed by a single circuit of the form (a, a), possibly with finite offshoots labelled $(a, \$)$ or $(\$, a)$.
3. A finite path followed by two circuits, one labelled (a, a) (possibly with finite offshoots) and one labelled $(a, \$)$, where both circuits have the same length q and for remainders $r_0, \ldots, r_{q-1}$ and some $0 < k < q-1$, we have that $\{a^{nq+r_0}, \ldots, a^{nq+r_k} : n \in \mathbb{N}_0\}$ are accepted by states on the $(a, \$)$ circuit and $\{a^{nq+r_{k+1}}, \ldots, a^{nq+r_{q-1}} : n \in \mathbb{N}_0\}$ are accepted by states on or offshoots of the (a, a) circuit.

Theorem

Monogenic subsemigroups of unary graph automatic semigroups are regular.

Using the structure of our automata we can show that a monogenic subsemigroup can be represented by finitely many arithmetic progressions, and so is a regular subsemigroup.

Theorem

Unary graph automatic semigroups are not periodic.

Sketch of proof.

Let λ be the lowest common multiple of the circuit lengths of our acceptor automata. We can find a product of generators $x_1 x_2 \dots x_n$, where a^{p_k} represents $x_1 x_2 \dots x_k$ for $1 \leq k \leq n$, such that $p_i = p_j \bmod \lambda$ with $p_j > p_i$ for some i and j. Then all powers of $x_{i+1} x_{i+2} \dots x_j$ are distinct. $\square$

Unary construcions

Let S and T be unary graph automatic semigroups. Then

- S^1 and S^0 are unary graph automatic.
- Any finitely generated subsemigroup of S of finite Rees index is unary graph automatic.
- Any finitely generated Rees matrix semigroup $M[S; I, J; P]$ is unary graph automatic.

Disjoint unions of the free monogenic semigroup are unary graph automatic.

Theorem

Let S and T be unary graph automatic, with T finite. Then if $S \times T$ finitely generated, it is unary graph automatic.

This does not hold if both S and T are infinite.
For example $\mathbb{N}_0 \times \mathbb{N}_0$ is not unary graph automatic.

Theorem

The free product of two semigroups is unary graph automatic if and only if both semigroups are trivial.

Open questions

If $S \times T$ is graph automatic, are S and T?
If $S * T$ is graph automatic, are S and T?
If the Rees matrix semigroup $M[S; I, J; P]$ is graph automatic, is S?

How do we show that a semigroup is not graph automatic?