Semigroup Graph Expansions
and their Green's Relations

Rebecca Noonan Heale

Heriot-Watt University, Edinburgh
and the Maxwell Institute for Mathematical Sciences
Supervised by Nick Gilbert

North British Semigroups and Applications Network, York
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Outline

1. Graph Expansions
   - History
   - Definitions
   - Graph Expansions
2. Green's Relations
   - $\mathcal{R}$ Relation
   - $\mathcal{L}$ Relation
3. Closing Remarks

Outline

Who's Who Guide to Graph Expansions

- Birget and Rhodes: fathers of semigroup expansions
- Margolis and Meakin: groups
- Gould and Gomes: monoids (right cancellative, unipotent)
- Elston: generalized graph expansions (via derived categories)
- Lawson, Margolis, and Steinberg: inverse semigroups
- Gilbert and Miller: ordered groupoids

Digraphs

Labeled digraph Γ: vertex set $V(\Gamma)$;
edge set $E(\Gamma)$;
edge label set X;
maps $\iota, \tau : E(\Gamma) \to V(\Gamma)$, $\ell : E(\Gamma) \to X$.

Labeled graph Γ: labeled digraph + inverse edges,
for all $e \in E(\Gamma)$, there exist $e^{-1} \in E(\Gamma)$
such that $e\iota = e^{-1}\tau$, $e\tau = e^{-1}\iota$,
if $e, f \in E(\Gamma)$ with $e\ell = f\ell$, then $e^{-1}\ell = f^{-1}\ell$.

Cayley Digraphs

	Groups	Semigroups
Presentation	$\langle X_G \rangle = G$ $f_G : X_G \to G$	$\langle X_S \rangle = S$ $f_S : X_S \to S$
Notation	$\mathrm{Cay}(G, X_G)$	$\mathrm{Cay}(S, X_S)$
Vertices	G	S
Edges	$\underset{a}{\bullet} \longrightarrow \underset{a(rf)}{\bullet}$ $r \in X_G \cup X_G^{-1}$	$\underset{a}{\bullet} \longrightarrow \underset{a(rf)}{\bullet}$ $r \in X_S$
Properties	labeled graph contains 1	labeled digraph

Group Graph Expansions (Margolis-Meakin, '89)

Start: $\mathrm{Cay}(G; X_G)$

Finish: $\mathcal{M}_{gp}(G, X_G)$

Elements: "Pieces" of the Cayley graph

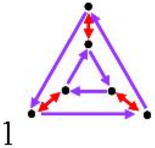
(P, c) = (subgraph, chosen vertex)

P is - finite,

- connected,

- $1, c \in V(P)$

Operation: $(P, a)(Q, b) = (P \cup aQ, ab)$



Group Graph Expansions (Margolis-Meakin, '89)

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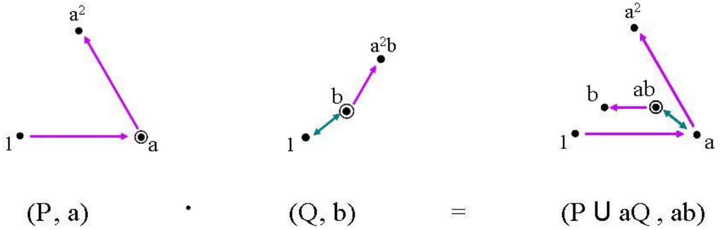


Semigroup Graph Expansions (RNH)

Start:	$\mathrm{Cay}(S; X_S)$
Finish:	$\mathcal{M}(S, X_S)$
Elements:	"Pieces" of the Cayley graph
	(r, P, c) = ("root", subgraph, chosen vertex)
	$r \in X_S$
	P is - finite,
	- rf-rooted,
	- $rf_S, c \in V(P)$
Operation:	$(r, P, c)(s, Q, d) = (r, P \cup cQ^1_s, cd)$

Semigroup Graph Expansions (RNH)

Start: $\mathrm{Cay}(S; X_S)$ Finish: $\mathcal{M}(S, X_S)$

Operation:

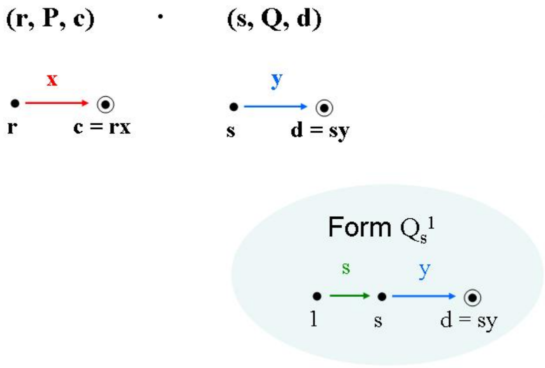
$(r, P, c) \cdot (s, Q, d)$

* For clarity, assume $rf = r$ and $sf = s$.

Semigroup Graph Expansions (RNH)

Start: $\mathrm{Cay}(S; X_S)$ Finish: $\mathcal{M}(S, X_S)$

Operation:



* For clarity, assume $rf = r$ and $sf = s$.

Semigroup Graph Expansions (RNH)

Start: $\mathrm{Cay}(S; X_S)$ Finish: $\mathcal{M}(S, X_S)$

Operation:

$$(\mathbf{r}, \mathbf{P}, \mathbf{c}) \cdot (\mathbf{s}, \mathbf{Q}, \mathbf{d}) = (\mathbf{r}, \mathbf{P} \cup \mathbf{c}\mathbf{Q}_{\mathbf{s}}^{\mathbf{1}}, \mathbf{cd})$$

$$\mathbf{r} \xrightarrow{\mathbf{x}} \mathbf{c} = \mathbf{rx} \qquad \mathbf{s} \xrightarrow{\mathbf{y}} \mathbf{d} = \mathbf{sy} \qquad \mathbf{r} \xrightarrow{\mathbf{x}} \mathbf{c} \xrightarrow{\mathbf{s}} \mathbf{cs} \xrightarrow{\mathbf{y}} \mathbf{cd} = \mathbf{rxsy}$$

Form Q_s^1

$$1 \xrightarrow{s} s \xrightarrow{y} d = sy$$

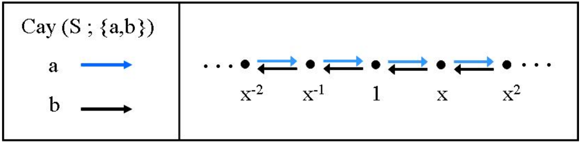
* For clarity, assume $rf = r$ and $sf = s$.

Example: Semigroup presentation of free group on one generator

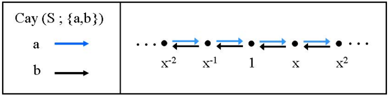
$G = \langle x | \emptyset \rangle;$

$S = \langle x,\ x^{-1} |\ xx^{-1} = x^{-1}x,\ x = xx^{-1}x,\ x^{-1} = x^{-1}xx^{-1} \rangle$

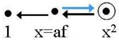
$X_S = \{a, b\}$, define $af_S = x$, $bf_S = x^{-1}$



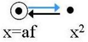
Example: Semigroup presentation of free group on one generator



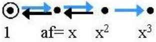
Sample elements: (a, P, x^2)



(a, Q, x)



$(a, R, 1)$



Idempotent!

Properties of Graph Expansions

$\mathcal{M}_{gp}(G, X_G)$

- Inverse monoid
- E-unitary
- Maximal group image G
- Generated by $X_G \cup X_G^{-1}$
- Residually finite, finite J-above, all subgroups are finite ...

$\mathcal{M}(S, X_S)$

- Semigroup
- E-dense iff S is E-dense
- If S is E-dense, has same maximal group image as S
- Finitely generated iff S is finite
- $c \in S$ periodic iff (r, P, c) periodic
- Residually finite, finite J-above, all subgroups are finite ...

Outline

$\mathcal{R}$ Relation

Definition: For $a, b \in S$, $a\mathcal{R}b \iff$ there exists $x, y \in S^1$ such that $ax = b$, $by = a$.

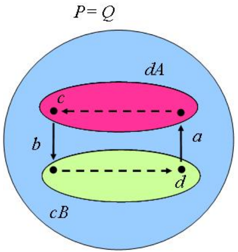
Prop. for $\mathcal{M}_{gp}(G, X_G)$: (M & M) $(P, c)\mathcal{R}(Q, d) \iff P = Q$.

Prop. for $\mathcal{M}(S, X_S)$: (RNH) $(r, P, c)\mathcal{R}(s, Q, d) \iff r = s$, $P = Q$, and there is a cycle in P containing c and d.

Picture for $\mathcal{R}$ Relation

- Let $(r, P, c)\mathcal{R}(s, Q, d)$.
- There exist (a, A, x) and (b, B, y) such that:

$$\begin{aligned}(r, P, c) &= (s, Q, d)(a, A, x) \\ &= (s, Q \cup dA^1_a, dx)\end{aligned} \qquad \begin{aligned}(s, Q, d) &= (r, P, c)(b, B, y) \\ &= (r, P \cup cB^1_b, cy)\end{aligned}$$



$\Longrightarrow$ $r = s$;

$\Longrightarrow$ $Q \subseteq P$ and $P \subseteq Q \Rightarrow P = Q$;

$\Longrightarrow$ Clearly a cycle connecting c and d.

$\mathcal{L}$ Relation

Definition: For $a, b \in S$, $a\mathcal{L}b \iff$ there exists $x, y \in S^1$ such that $xa = b$, $yb = a$.

Prop. for $\mathcal{M}_{gp}(G, X_G)$: (M & M) $(P, c)\mathcal{L}(Q, d) \iff c^{-1}P = d^{-1}Q$.

Prop. for $\mathcal{M}(S, X_S)$: (RNH) $(r, P, c)\mathcal{L}(s, Q, d) \iff$ there exist $a, b \in S$ such that:

(a) $ac = c$ and $bd = d$;
(b) $aP_r^1 \subseteq P$ and $bQ_s^1 \subseteq Q$;
(c) aP_r^1 is isomorphic to bQ_s^1 (as labeled graphs) and this isomorphism sends c to d.

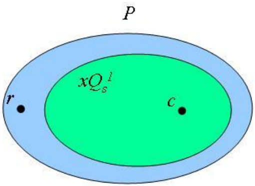
Idea Behind $\mathcal{L}$ Relation $\Rightarrow$

- Let $(r, P, c)\mathcal{L}(s, Q, d)$.
- There exist (r, A, x) and (s, B, y) such that:

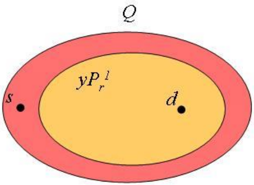
$$\begin{aligned}(r, P, c) &= (r, A, x)(s, Q, d) \\ &= (r, A \cup xQ_s^1, xd)\end{aligned} \qquad \begin{aligned}(s, Q, d) &= (s, B, y)(r, P, c) \\ &= (s, B \cup yP_r^1, yc)\end{aligned}$$

$$\Longrightarrow \ xQ_s^1 \subseteq P,\ yP_r^1 \subseteq Q.$$

Idea Behind $\mathcal{L}$ Relation $\Rightarrow$

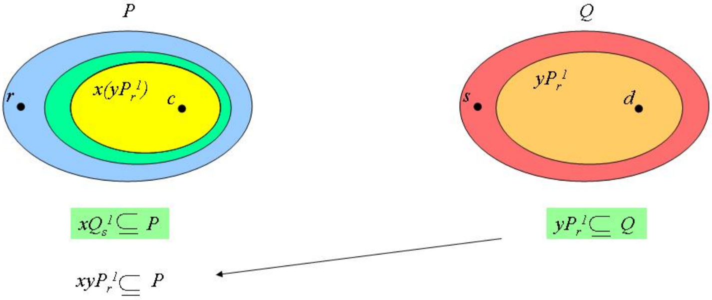


$xQ_s^1 \subseteq P$

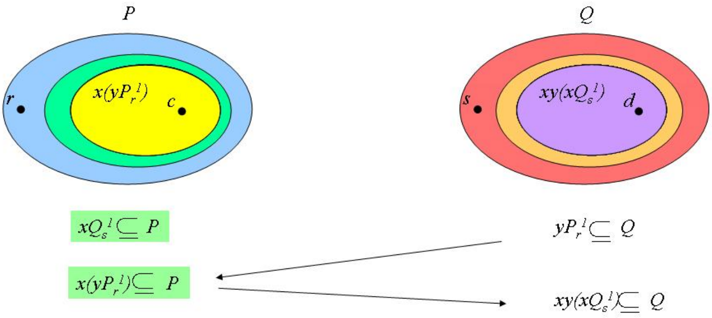


$yP_r^1 \subseteq Q$

Idea Behind $\mathcal{L}$ Relation $\Rightarrow$



Idea Behind $\mathcal{L}$ Relation $\Rightarrow$



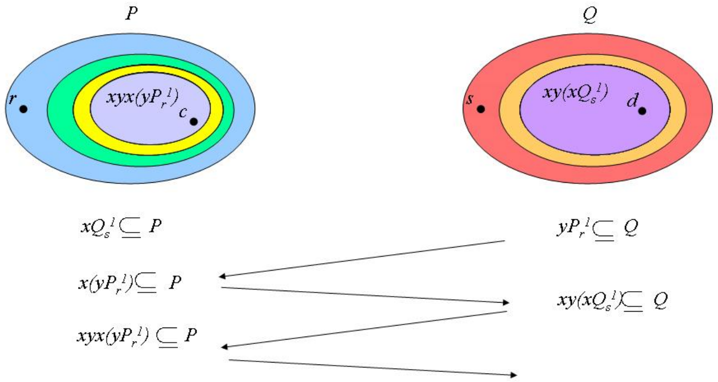
Idea Behind $\mathcal{L}$ Relation $\Rightarrow$

$xQ_s^1 \subseteq P$ $\qquad$ $\boxed{yP_r^1 \subseteq Q}$

$x(yP_r^1) \subseteq P$ $\qquad$ $\boxed{xy(xQ_s^1) \subseteq Q}$

$xyx(yP_r^1) \subseteq P$

Idea Behind $\mathcal{L}$ Relation $\Rightarrow$



Idea Behind $\mathcal{L}$ Relation $\Rightarrow$

$xQ_s^1 \subseteq P$

$yP_r^1 \subseteq Q$

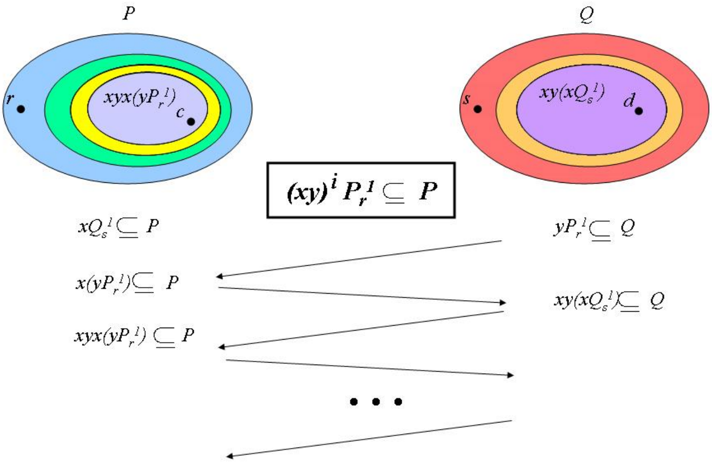
$x(yP_r^1) \subseteq P$

$xy(xQ_s^1) \subseteq Q$

$xyx(yP_r^1) \subseteq P$

$\cdots$

Idea Behind $\mathcal{L}$ Relation $\Rightarrow$



Idea Behind $\mathcal{L}$ Relation $\Rightarrow$

- See that $(xy)^i \in V(P)$ for all $i \in \mathbb{N}$.
- Since P is a finite graph, (xy) is periodic
- There exists a smallest $k, m \in \mathbb{N}$ s.t. $(xy)^k = (xy)^{k+m}$

Idea Behind $\mathcal{L}$ Relation $\Rightarrow$

- Recall the **Proposition** for $\mathcal{M}(S, X_S)$:

  $(r, P, c)\mathcal{L}(s, Q, d) \iff$ *there exist* $a, b \in S$ *s.t.:*

  (a) $ac = c$ *and* $bd = d$*;*

  (b) $aP_r^1 \subseteq P$ *and* $bQ_s^1 \subseteq Q$*;*

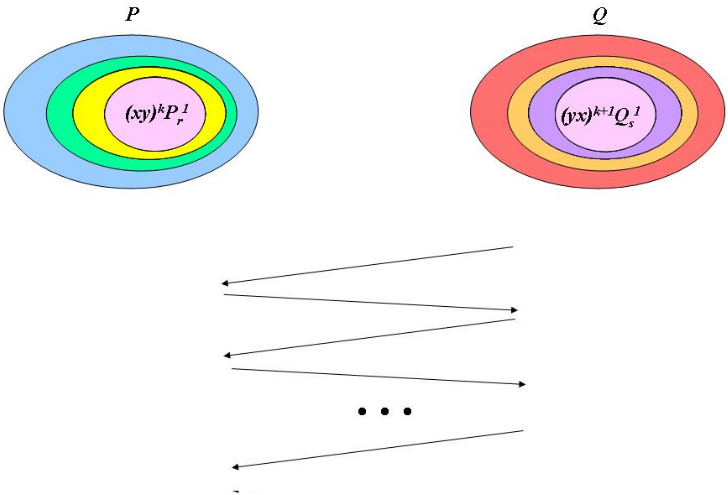
  (c) *there exists a label-preserving isomorphism* $\theta : aP_r^1 \to bQ_s^1$ *such that* $c\theta = d$*.*

- Use $a = (xy)^k$, $b = (yx)^{k+1}$.
- Already have

  (a) $(xy)^k P \subseteq P$,

  (b) $(xy)c = c \implies (xy)^k c = c$

Idea Behind $\mathcal{L}$ Relation $\Rightarrow$



Idea Behind $\mathcal{L}$ Relation $\Rightarrow$

- Therefore aP_r^1 is isomorphic to bQ_s^1 (as labeled graphs) and this isomorphism sends c to d.

Idea Behind $\mathcal{L}$ Relation $\Leftarrow$

- Let $(r, P, c)\mathcal{L}(s, Q, d)$.
- There are elements a, b such that:

  (a) *$ac = c$ and $bd = d$;*
  (b) *$aP_r^1 \subseteq P$ and $bQ_s^1 \subseteq Q$;*
  (c) *aP_r^1 is isomorphic to bQ_s^1 (as labeled graphs) and this isomorphism sends c to d.*

- Let $\theta : aP_r^1 \to bQ_s^1$ be the isomorphism.

Idea Behind $\mathcal{L}$ Relation $\Leftarrow$

- Notice $(aP_r^1)\theta = bQ_s^1$.
- Need two results (Obtained from "rootedness" of P and determinism of Cayley graph):

  1. $a\theta(P_r^1) = (aP_r^1)\theta$;
  2. $(a\theta)c = (ac)\theta = c\theta = d$

- Therefore: $(s, Q, a\theta)(r, P, c)$ = $(s, Q \cup (a\theta)P_r^1, (a\theta)c)$
  (s, Q, d)
- A little more work $\ldots$ $(r, P, c)\mathcal{L}(s, Q, d)$.

Examples of $\mathcal{L}$ Relation

- Suppose S is a group (generated as a semigroup).
- Use Prop: $(r, P, c)\mathcal{L}(s, Q, d) \Leftrightarrow$ there exist $a, b \in S$ s.t.:
  (a) $ac = c$ and $bd = d$;
  (b) $aP_r^1 \subseteq P$ and $bQ_s^1 \subseteq Q$;
  (c) aP_r^1 is isomorphic to bQ_s^1 (with c sent to d).
- This implies
  $\Rightarrow$ $a = b = 1$
  $\Rightarrow$ $P_r^1 = P$ and $Q_s^1 = Q$
  $\Rightarrow$ $c^{-1}P = d^{-1}Q$

Examples of $\mathcal{L}$ Relation

- Suppose S is a semilattice.
- Use Prop: $(r, P, c)\mathcal{L}(s, Q, d) \Leftrightarrow$ there exist $a, b \in S$ s.t.:
  (a) $ac = c$ and $bd = d$;
  (b) $aP_r^1 \subseteq P$ and $bQ_s^1 \subseteq Q$;
  (c) aP_r^1 is isomorphic to bQ_s^1 (with c sent to d).
- This implies

$\Rightarrow \; c = d \qquad\qquad \Rightarrow \; a = b$

$\Rightarrow \; c \leq a \qquad\qquad \Rightarrow \; aP_r^1 = aQ_s^1$

Outline

Understanding Generalizations of the Green's Relations - Using Graph Expansion Techniques

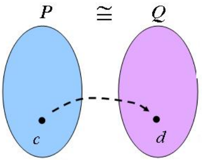
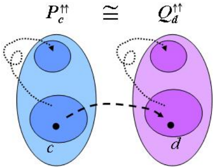
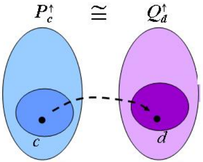
Switch to graph expansions of a MONOID T;

- $\mathcal{L}$;
- Star Relation: $a\mathcal{L}^*b \Leftrightarrow$ for all $x, y \in T$, $ax = ay$ if and only if $bx = by$
- Tilde Relation: $a\tilde{\mathcal{L}}b$: if and only if a and b have the same idempotent right identities, i.e.

$$\{c|ac = a \text{ and } c^2 = c\} = \{d|bd = b \text{ and } d^2 = d\}.$$

Generalizations of Green's Relations

Left Cancellative Monoids

$\mathcal{L}$	$(P,c)\mathcal{L}^*(Q,d)$ $(P,c)(X,x) = (P,c)(Y,y) \Leftrightarrow$ $(Q,d)(X,x) = (Q,d)(Y,y)$	$(P,c)\tilde{\mathcal{L}}(Q,d)$ $(P,c)(X,x) = (P,c) \Leftrightarrow$ $(Q,d)(X,x) = (Q,d)$ (X,x) is idempotent
		

Köszönöm!

Obrigada!

Thank you!

Muchas Gracias!

Vielen Dank!