Khovanov's Presheaf on Some Ordered Groupoids

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14th January, 2015

Keywords

Groupoid, Ordered Groupoid and Category

Space of operation

Definition

A groupoid G is a set equipped with the operation $G^2 \to G; (x, y) \mapsto xy$ where $G^2 \subset G \times G$ called composable pairs and an inverse map $G \to G; x \mapsto x^{-1}$ satisfying the following conditions

- $(x^{-1})^{-1} = x$
- $(x, y), (y, z) \in G^2$ if there exist $(xy, z), (x, yz) \in G^2$ implies $(xy)z = x(yz)$
- $(x^{-1}, x), (x, y) \in G^2$ then $x^{-1}(xy) = y$
- $(x, x^{-1}), (z, x) \in G^2$ then $(zx^{-1})x = z$

Notation

$x\mathbf{d} = xx^{-1}$ and $x\mathbf{r} = x^{-1}x$ for $x \in G$. Denote by $G_0 = \{x : x\mathbf{d} = x\mathbf{r} = x\}$

Platform for the work

A groupoid G together with a natural partial order $\leq$ is called an *ordered* groupoid if it accounts for

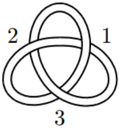
- $x \leq y \;\Rightarrow x^{-1} \leq y^{-1}$
- for $x \leq y,\; u \leq v$ if $\exists\; xu,\; \exists\; yv \;\Rightarrow xu \leq yv$
- if $x \in G,\; e \in G_0$ and $e \leq x\mathbf{d}$ then $\exists$ a unique element $(x|e)$ called the *restriction* of x to e such that $(x|e)\mathbf{d} = e$ and $(x|e) \leq x$.
- if $x \in G,\; e \in G_0$ and $e \leq x\mathbf{r}$ then $\exists$ a unique element $(e|x)$ called the *corestriction* of x to e such that $(e|x)\mathbf{r} = e$ and $(e|x) \leq x$

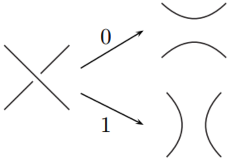
Examples of Ordered Groupoids

- Groups are ordered groupoids with equality as the natural partial order
- Posets
- For a group G and a poset E, then $G \times E$ is an ordered groupoid with $(g, e)(g', e') = (gg', e)$ whenever $e = e'$ and $(g, e) \leq (g', e')$ iff $g = g'$ and $e \leq e'$.

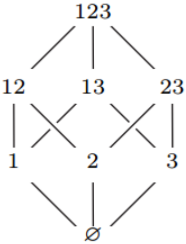
Further identification of ordered groupoids

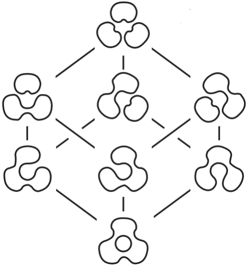
Consider a link diagram. Each crossing can be 0- or 1-resolved.





Complete resolution of subsets of the crossings identifies a Boolean lattice

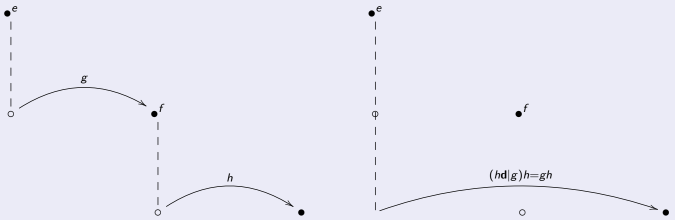




Loganathan's category

The category $\mathcal{L}(G)$ consist of the following data;

- $(\mathcal{L}(G))_0 = G_0$
- $\mathcal{L}(G) = \{(e,g) \in G_0 \times G : g\mathbf{d} = g^{-1}g \leq e\}$
  - $(e,g)\mathbf{d} = e$ and $(e,g)\mathbf{r} = g\mathbf{r}$.
  - $(e,g)(f,h) = (e,gh)$ when $g\mathbf{r} = f$



Remark

- $\mathcal{L}(G)$ is left cancellative
  $(e,g)(f,h) = (e,g)(k,l) \Rightarrow (f,h) = (k,l)$
- each morphism (e,g) uniquely decomposable, $(e, gg^{-1})(gg^{-1}, g)$
- $\mathcal{L}(G)$ is a Zappa-Szép product of the categories G_0 and G

Modules for $\mathcal{L}(G)$

An $\mathcal{L}(G)$-module is an functor $\mathcal{L}(G) \to \mathbf{Ab}$.

- $x \mapsto M_x$ for all $x \in G_0$
- a homomorphism $M_x \to M_y$ whenever $y \leq x$
- an isomorphism $M_{xx^{-1}} \to M_{x^{-1}x}$

The trivial or constant $\mathcal{L}(G)$-module $\Delta : \mathbf{Ab} \to \mathrm{Mod}(\mathcal{L}(G))$ is identified with

- $x \mapsto \Delta B_x = B$
- $\mathbf{1} : \Delta B_x \to \Delta B_y$ whenever $y \leq x$

The category $\mathrm{Mod}(\mathcal{L}(G))$

- has objects $\mathcal{L}(G)$-modules and natural transformations as morphisms
- is an abelian category
- has enough injectives and projectives.

Khovanov's $\mathcal{L}(G)$-module

Let ordered groupoid, G be the boolean lattice associated to a link diagram.

The rank two free abelian group $V = \mathbb{Z}[1, u]$ becomes a frobenius algebra using the maps
$m : V \otimes V \to V; 1 \otimes 1 \mapsto 1, 1 \otimes u \mapsto u, u \otimes u \mapsto 0$
$\epsilon : V \to \mathbb{Z}; 1 \mapsto 0, u \mapsto 1$
$\Delta : V \to V \otimes V; 1 \mapsto 1 \otimes u + u \otimes 1, u \mapsto u \otimes u$

Khovanov's presheaf functor on the cubes defines an $\mathcal{L}(G)$-module $F_{KH} : G \to \mathbf{Ab}; x \mapsto V^{\otimes k}$

Cohomology of $\mathcal{L}(G)$

The inverse limit functor $\varprojlim : \mathrm{Mod}(\mathcal{L}(G)) \to \mathbf{Ab}$ is right adjoint to the exact Δ functor.

- $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{L}(G))}(\Delta A, M) \cong \mathrm{Hom}_{\mathbf{Ab}}(A, \varprojlim M)$ for $A \in \mathbf{Ab}$ and $M \in \mathrm{Mod}(\mathcal{L}(G))$.
- it is left exact
  for every $0 \to M \to M' \to M''$ the sequence
  $0 \to \varprojlim M \to \varprojlim M' \to \varprojlim M''$ is exact.
- it has right derived functors

The *nth* cohomology of $\mathcal{L}(G)$ with coefficient in the module M is defined by
$\mathrm{H}^n(\mathcal{L}(G), M) = \varprojlim^i_{\mathcal{L}(G)} M \cong R^i(\mathrm{Hom}_{\mathcal{L}(G))}(P_*, M))$ where P_* is a projective resolution of $\Delta\mathbb{Z}$.

Theorem

Let D be the associated link diagram of the ordered groupoid G and Khovanov's $\mathcal{L}(G)$-module $F_{KH} : G \to \mathbf{Ab}$. Then Khovanov's homological link invariant is given by

$$\mathrm{H}^n_{KH}(\mathcal{L}(G), F_{KH}) = \varprojlim^i_{\mathcal{L}(G)} F_{KH} \cong R^i(\mathrm{Hom}_{\mathcal{L}(G))}(P_*, F_{KH}))$$

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THANK YOU