Free idempotent generated semigroups and endomorphism monoids of free G-acts

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Based on joint work with Igor Dolinka and Victoria Gould

Free idempotent generated semigroups

Let E be a biordered set (equivalently, a set of idempotents E of a semigroup S).

The free idempotent generated semigroup $\mathsf{IG}(E)$ is a free object in the category of semigroups that are generated by E, defined by

$$\mathsf{IG}(E) = \langle \overline{E} : \bar{e}\bar{f} = \overline{ef},\ e, f \in E, \{e, f\} \cap \{ef, fe\} \neq \emptyset \rangle.$$

where $\overline{E} = \{\bar{e} : e \in E\}$.

Note It is more usual to identify elements of E with those of $\overline{E}$, but it helps the clarity of our later arguments to make this distinction.

Facts

1. $\mathrm{IG}(E) = \langle \overline{E} \rangle$.
2. The natural map $\phi : \mathrm{IG}(E) \to S$, given by $\bar{e}\phi = e$, is a morphism onto $S' = \langle E(S) \rangle$.
3. The restriction of ϕ to the set of idempotents of $\mathrm{IG}(E)$ is a bijection.
4. The morphism ϕ induces a bijection between the set of all $\mathcal{R}$-classes (resp. $\mathcal{L}$-classes) in the $\mathcal{D}$-class of $\bar{e}$ in $\mathrm{IG}(E)$ and the corresponding set in $S' = \langle E(S) \rangle$.
5. The morphism ϕ is an onto morphism from $H_{\bar{e}}$ to H_e.

Maximal subgroups of IG(E)

Work of Pastijn (1977, 1980), Nambooripad and Pastijn (1980), McElwee (2002) led to a conjecture that all these groups must be free groups.

Brittenham, Margolis and Meakin (2009)

$\mathbb{Z} \oplus \mathbb{Z}$ can be a maximal subgroup of IG(E), for some E.

Gray and Ruskuc (2012)

Any group occurs as a maximal subgroup of some IG(E), a general presentation and a special choice of E are needed.

Gould and Yang (2012)

Any group occurs as a maximal subgroup of a natural IG(E), a simple approach suffices.

Dolinka and Ruskuc (2013)

Any group occurs as IG(E) for some *band.*

Maximal subgroups of IG(E)

Given a special biordered set E, which kind of groups can be the maximal subgroups of IG(E)?

Let S be a semigroup with $E = E(S)$. Let $e \in E$. Our aim is to find the relationship between the maximal subgroup $H_{\overline{e}}$ of IG(E) with identity $\overline{e}$ and the maximal subgroup H_e of S with identity e.

There is an onto morphism from $H_{\overline{e}}$ to H_e.

Is $H_{\overline{e}} \cong H_e$, for some E and some $e \in E$?

Full transformation monoids and full linear monoids

$\mathcal{T}_n$ ($\mathcal{PT}_n$) - full (partial) transformation monoid, E - its biordered set.

Gray and Ruskuc (2012); Dolinka (2013)

$\operatorname{rank} e = r < n-1$, $H_{\overline{e}} \cong H_e \cong \mathcal{S}_r$.

Brittenham, Margolis and Meakin (2010)

$M_n(D)$ - full linear monoid, E - its biordered set.

$\operatorname{rank} e = 1$ and $n \geq 3$, $H_{\overline{e}} \cong H_e \cong D^*$.

Dolinka and Gray (2012)

$\operatorname{rank} e = r < n/3$ and $n \geq 4$, $H_{\overline{e}} \cong H_e \cong GL_r(D)$.

Note $\operatorname{rank} e = n-1$, $H_{\overline{e}}$ is free; $\operatorname{rank} e = n$, $H_{\overline{e}}$ is trivial.

Sets and vector spaces over division rings are examples of **independence algebras.**

Fountain and Lewin (1992)

Let $\mathbf{A}$ be an independence algebra of rank n, where $n \in \mathbb{N}$ is finite. Let $\operatorname{End}\mathbf{A}$ be the endomorpshim monoid of $\mathbf{A}$. Then

$$S(\operatorname{End}\mathbf{A}) = \{\alpha \in \operatorname{End}\mathbf{A} : \operatorname{rank}\alpha < n\} = \langle E \setminus \{I\}\rangle.$$

Gould (1995)

For any $\alpha, \beta \in \operatorname{End}\mathbf{A}$, we have the following:

(i) $\operatorname{im}\alpha = \operatorname{im}\beta$ if and only if $\alpha\,\mathcal{L}\,\beta$;

(ii) $\ker\alpha = \ker\beta$ if and only if $\alpha\,\mathcal{R}\,\beta$;

(iii) $\operatorname{rank}\alpha = \operatorname{rank}\beta$ if and only if $\alpha\,\mathcal{D}\,\beta$ if and only if $\alpha\,\mathcal{J}\,\beta$.

Independence algebras

The results on the biordered set of idempotents of $\mathcal{T}_n$ and $M_n(D)$ suggest that it would be worth looking into the maximal subgroups of $\mathrm{IG}(E)$, where $E = E(\mathrm{End}\,\mathbf{A})$.

The diverse method needed in the biordered sets of $\mathcal{T}_n$ and $M_n(D)$ indicate that it would be very hard to find a unified approach to $\mathrm{End}\,\mathbf{A}$.

It was pointed out by Gould that **free** G-**acts** provide us with another kind of independence algebras.

Free G-acts

Let G be a group, $n \in \mathbb{N}, n \geq 3$. Let $F_n(G)$ be a rank n **free left G-act.**

Recall that, as a set,

$$F_n(G) = \{gx_i : g \in G, i \in [1, n]\};$$

identify x_i with $1x_i$, where 1 is the identity of G;

$gx_i = hx_j$ if and only if $g = h$ and $i = j$;

the action of G is given by $g(hx_i) = (gh)x_i$.

Endomorphism monoids of free left G-acts

Let End $F_n(G)$ be the endomorphism monoid of $F_n(G)$ with $E = E(\text{End}\, F_n(G))$.

The **rank** of an element of End $F_n(G)$ is the minimal number of (free) generators in its image.

An element $\alpha \in \text{End}\, F_n(G)$ depends only on its action on the free generators $\{x_i : i \in [1, n]\}$.

For convenience we denote α by

$$\alpha = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ w_1^\alpha x_{1\overline{\alpha}} & w_2^\alpha x_{2\overline{\alpha}} & \dots & w_n^\alpha x_{n\overline{\alpha}} \end{pmatrix},$$

where $\overline{\alpha} \in \mathcal{T}_n$, $w_1^\alpha, \cdots, w_n^\alpha \in G$.

Note End $F_n(G) \cong G \wr \mathcal{S}_n$ and $\mathcal{S}(\text{End}\, F_n(G)) = \langle E \setminus \{I\} \rangle$.

Endomorphism monoids of free left G-acts

For any rank r idempotent $\varepsilon \in E$, where $1 \leq r \leq n$, we have

$$H_\varepsilon \cong G \wr \mathcal{S}_r.$$

How about the maximal subgroup $H_{\overline{\varepsilon}}$ of $\mathrm{IG}(E)$?

A presentation for $H_{\overline{\varepsilon}}$

To specialise Gray and Ruškuc's presentation of maximal subgroups of $\mathrm{IG}(E)$ to our particular circumstance.

Step 1

To obtain an explicit description of a Rees matrix semigroup isomorphic to the semigroup $D_r^0 = D_r \cup \{0\}$, where

$$D_r = \{\alpha \in \mathrm{End}\, F_n(G) \mid \text{rank } \alpha = r\}.$$

Let I and Λ denote the set of $\mathcal{R}$-classes and the set of $\mathcal{L}$-classes of D_r, respectively.

Here we may take I as the set of kernels of elements in D_r, and $\Lambda = \{(u_1, u_2, \ldots, u_r) : 1 \leq u_1 < u_2 < \ldots < u_r \leq n\} \subseteq [1, n]^r$.

Let $H_{i\lambda} = R_i \cap L_\lambda$.

A presentation for $H_{\overline{\varepsilon}}$

Assume $1 \in I \cap \Lambda$ with

$$1 = \langle (x_1, x_i) : r+1 \leq i \leq n \rangle \in I, 1 = (1, \cdots, r) \in \Lambda.$$

So $H = H_{11}$ is a group with identity $\varepsilon = \varepsilon_{11}$.

A typical element of H looks like

$$\alpha = \begin{pmatrix} x_1 & x_2 & \dots & x_r & x_{r+1} & \dots & x_n \\ w_1^\alpha x_{1\overline{\alpha}} & w_2^\alpha x_{2\overline{\alpha}} & \dots & w_r^\alpha x_{r\overline{\alpha}} & w_1^\alpha x_{1\overline{\alpha}} & \dots & w_1^\alpha x_{1\overline{\alpha}} \end{pmatrix}$$

where $\overline{\alpha} \in \mathcal{T}_n$, $w_1^\alpha, \cdots, w_r^\alpha \in G$.

Abbreviate α as

$$\alpha = \begin{pmatrix} x_1 & x_2 & \dots & x_r \\ w_1^\alpha x_{1\overline{\alpha}} & w_2^\alpha x_{2\overline{\alpha}} & \dots & w_r^\alpha x_{r\overline{\alpha}} \end{pmatrix}.$$

In particular,

$$\varepsilon = \varepsilon_{11} = \begin{pmatrix} x_1 & x_2 & \dots & x_r \\ x_1 & x_2 & \dots & x_r \end{pmatrix}.$$

A presentation for $H_{\overline{\mathcal{E}}}$

For any $\alpha \in D_r$, $\ker \overline{\alpha}$ induces a partition

$$\{B_1^\alpha, \cdots, B_r^\alpha\}$$

on $[1, n]$ with a set of minimum elements

$$l_1^\alpha, \cdots, l_r^\alpha \text{ such that } l_1^\alpha < \cdots < l_r^\alpha.$$

Put

$$\Theta = \{\alpha \in D_r : x_{l_j^\alpha}\alpha = x_j, j \in [1, r]\}.$$

Then it is a transversal of the $\mathcal{H}$-classes of L_1.

For each $i \in I$, define $\mathbf{r}_i$ as the unique element in $\Theta \cap H_{i1}$.

We say that $\mathbf{r}_i$ lies in **district** $(l_1^{\mathbf{r}_i}, l_2^{\mathbf{r}_i}, \cdots, l_r^{\mathbf{r}_i})$ (of course, $1 = l_1^{\mathbf{r}_i}$).

For each $\lambda = (u_1, u_2, \ldots, u_r) \in \Lambda$, define

$$\mathbf{q}_\lambda = \mathbf{q}_{(u_1, \cdots, u_r)} = \begin{pmatrix} x_1 & x_2 & \cdots & x_r & x_{r+1} & \cdots & x_n \\ x_{u_1} & x_{u_2} & \cdots & x_{u_r} & x_{u_1} & \cdots & x_{u_1} \end{pmatrix}.$$

A presentation for $H_{\overline{\mathcal{E}}}$

We have that $D_r^0 = D_r \cup \{0\}$ is completely 0-simple, and hence

$$D_r^0 \cong \mathcal{M}^0(H; I, \Lambda; P),$$

where $P = (\mathbf{p}_{\lambda i})$ and

$$\mathbf{p}_{\lambda i} = (\mathbf{q}_\lambda \mathbf{r}_i) \text{ if } \operatorname{rank} \mathbf{q}_\lambda \mathbf{r}_i = r$$

and is 0 else.

Note

$$\begin{bmatrix} \varepsilon_{i\lambda} & \varepsilon_{i\mu} \\ \varepsilon_{k\lambda} & \varepsilon_{k\mu} \end{bmatrix} \text{ is a singular square} \Longleftrightarrow \mathbf{p}_{\lambda i}^{-1}\mathbf{p}_{\lambda k} = \mathbf{p}_{\mu i}^{-1}\mathbf{p}_{\mu k}.$$

Step 2

Define a **schreier system** of words $\{\mathbf{h}_\lambda : \lambda \in \Lambda\}$ inductively, using the restriction of the lexicographic order on $[1, n]^r$ to Λ.

Put $\mathbf{h}_{(1,2,\cdots,r)} = 1$;

For any $(u_1, u_2, \ldots, u_r) > (1, 2, \cdots, r)$, take $u_0 = 0$ and i the largest such that $u_i - u_{i-1} > 1$. Then

$$(u_1, \ldots, u_{i-1}, u_i - 1, u_{i+1}, \ldots, u_r) < (u_1, u_2, \ldots, u_r).$$

Define

$$\mathbf{h}_{(u_1,\cdots,u_r)} = \mathbf{h}_{(u_1,\cdots,u_{i-1},u_i-1,u_{i+1},\cdots,u_r)}\alpha_{(u_1,\cdots,u_r)},$$

where

$$\alpha_{(u_1,\cdots,u_r)} = \begin{pmatrix} x_1 & \cdots & x_{u_1} & x_{u_1+1} & \cdots & x_{u_2} & \cdots & x_{u_{r-1}+1} & \cdots & x_{u_r} & x_{u_r+1} & \cdots & x_n \\ x_{u_1} & \cdots & x_{u_1} & x_{u_2} & \cdots & x_{u_2} & \cdots & x_{u_r} & \cdots & x_{u_r} & x_{u_r} & \cdots & x_{u_r} \end{pmatrix}$$

Facts

1. $\varepsilon \mathbf{h}_{(u_1,\ldots,u_r)} = \mathbf{q}_{(u_1,\ldots,u_r)}$.
2. $\mathbf{h}_{(u_1,\cdots,u_r)}$ induces a bijection from $L_{(1,\cdots,r)}$ onto $L_{(u_1,\cdots,u_r)}$ in both End $F_n(G)$ and IG(E).

Hence $\{\mathbf{h}_\lambda : \lambda \in \Lambda\}$ forms the required schreier system for the presentation for $\overline{H} = H_{\overline{\varepsilon}}$.

Step 3

Define a function

$$\omega : I \longrightarrow \Lambda, i \mapsto \omega(i) = (l_1^{\mathbf{r}_i}, l_2^{\mathbf{r}_i}, \ldots, l_r^{\mathbf{r}_i}).$$

Note $\mathbf{p}_{\omega(i),i} = \varepsilon$.

Put

$$K = \{(i, \lambda) \in I \times \Lambda : H_{i\lambda} \text{ is a group}\}.$$

Proposition Let $E = E(\mathrm{End}\, F_n(G))$. Then the maximal subgroup $\overline{H}$ of $\overline{\varepsilon}$ in $\mathrm{IG}(E)$ is defined by the presentation

$$\mathcal{P} = \langle F : \Sigma \rangle$$

with generators:

$$F = \{f_{i,\lambda} : \quad (i,\lambda) \in K\}$$

and defining relations Σ:
$(R1)$ $f_{i,\lambda} = f_{i,\mu} \quad (\mathbf{h}_{\lambda}\varepsilon_{i\mu} = \mathbf{h}_{\mu})$;
$(R2)$ $f_{i,\omega(i)} = 1 \quad (i \in I)$;
$(R3)$ $f_{i,\lambda}^{-1} f_{i,\mu} = f_{k,\lambda}^{-1} f_{k,\mu} \quad \left(\begin{bmatrix} \varepsilon_{i\lambda} & \varepsilon_{i\mu} \\ \varepsilon_{k\lambda} & \varepsilon_{k\mu} \end{bmatrix} \text{ is singular i.e. } \mathbf{p}_{\lambda i}^{-1}\mathbf{p}_{\lambda k} = \mathbf{p}_{\mu i}^{-1}\mathbf{p}_{\mu k} \right)$.

Note If $\operatorname{rank}\varepsilon = n-1$, then $H_{\overline{\varepsilon}}$ is free, as no non-trivial singular squares exist; if $\operatorname{rank}\varepsilon = n$, then $H_{\overline{\varepsilon}}$ is trivial.

How about $H_{\overline{\varepsilon}}$, where $1 \leq \operatorname{rank}\varepsilon \leq n-2$?

A presentation for $H_{\overline{\mathcal{E}}}$

Given a pair $(i, \lambda) \in K$, we have a generator $f_{i,\lambda}$ and an element $0 \neq \mathbf{p}_{\lambda i} \in P$.

To find the relationship between these generators $f_{i,\lambda}$ and non-zero elements $\mathbf{p}_{\lambda i} \in P$.

Identity generators

Lemma If $(i, \lambda) \in K$ and $\mathbf{p}_{\lambda i} = \varepsilon$, then $f_{i,\lambda} = 1_{\overline{H}}$.

Idea. The proof follows by induction on $\lambda \in \Lambda$, ordered lexicographically. Here we make use of our particular choice of schreier system and function ω.

Generators corresponding to the same columns or rows

Lemma If $\mathbf{p}_{\lambda i} = \mathbf{p}_{\mu i}$, then $f_{i,\lambda} = f_{i,\mu}$.

The proof is straightforward.

Lemma If $\mathbf{p}_{\lambda i} = \mathbf{p}_{\lambda j}$, then $f_{i,\lambda} = f_{j,\lambda}$.

Idea. For any $i, j \in I$, suppose that $\mathbf{r}_i$ and $\mathbf{r}_j$ lie in districts $(1, k_2, \cdots, k_r)$ and $(1, l_2, \cdots, l_r)$, respectively. We call $u \in [1, n]$ a mutually **bad** element of $\mathbf{r}_i$ with respect to $\mathbf{r}_j$, if there exist $m, s \in [1, r]$ such that $u = k_m = l_s$, but $m \neq s$; all other elements are said to be mutually **good** with respect to $\mathbf{r}_i$ and $\mathbf{r}_j$.

We proceed by induction on the number of bad elements.

Connectivity of elements in the sandwich matrix

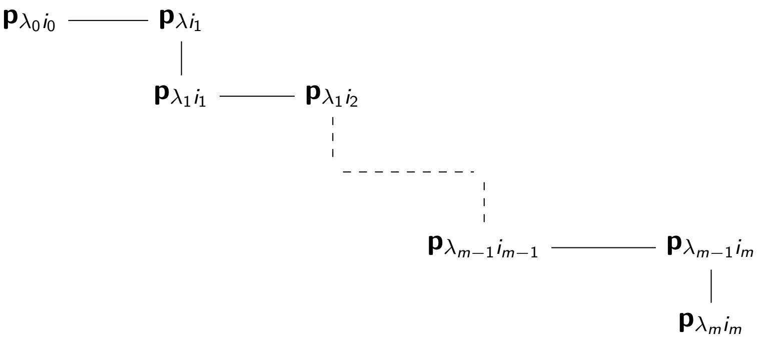
Definition Let $i, j \in I$ and $\lambda, \mu \in \Lambda$ such that $\mathbf{p}_{\lambda i} = \mathbf{p}_{\mu j}$.
We say that $(i, \lambda), (j, \mu)$ are *connected* if there exist

$$i = i_0, i_1, \ldots, i_m = j \in I \text{ and } \lambda = \lambda_0, \lambda_1, \ldots, \lambda_m = \mu \in \Lambda$$

such that for $0 \leq k < m$ we have $\mathbf{p}_{\lambda_k i_k} = \mathbf{p}_{\lambda_k, i_{k+1}} = \mathbf{p}_{\lambda_{k+1} i_{k+1}}$.

Connectivity of elements in the sandwich matrix

The following picture illustrates that $(i, \lambda) = (i_0, \lambda_0)$ is connected to $(j, \mu) = (i_m, \lambda_m)$:



Lemma Let $i, j \in I$ and $\lambda, \mu \in \Lambda$ be such that $\mathbf{p}_{\lambda i} = \mathbf{p}_{\mu j}$ where $(i, \lambda), (j, \mu)$ are connected. Then $f_{i,\lambda} = f_{j,\mu}$.

The result for $n \geq 2r+1$

Lemma Let $n \geq 2r+1$. Let $\lambda = (u_1, \cdots, u_r) \in \Lambda$, and $i \in I$ with $\mathbf{p}_{\lambda i} \in H$. Then (i, λ) is connected to (j, μ) for some $j \in I$ and $\mu = (n-r+1, \cdots, n)$.

Consequently, if $\mathbf{p}_{\lambda i} = \mathbf{p}_{\nu k}$ for any $i, k \in I$ and $\lambda, \nu \in \Lambda$, then $f_{i,\lambda} = f_{k,\nu}$.

We may define

$$f_\phi = f_{i,\lambda}, \text{if } \mathbf{p}_{\lambda i} = \phi \in H.$$

Lemma Let $r \leq n/3$. Then for any $\phi, \theta \in H$,

$$f_{\phi\theta} = f_\theta f_\phi \text{ and } f_{\phi^{-1}} = f_\phi^{-1}$$

Note Every element of H appears in P.

Theorem Let $r \leq n/3$. Then

$$\overline{H} \cong H, \;\; f_\phi \mapsto \phi^{-1}.$$

The result for $r \leq n-2$

For larger r this strategy will fail... :-(

Two main problems:

for $r \geq n/2$, not every element of H lies in P;

we lose connectivity of elements in P, even if $r = n/2$.

However, for $r \leq n-2$ all elements with **simple form**

$$\phi = \begin{pmatrix} x_1 & x_2 & \cdots & x_{k-1} & x_k & x_{k+1} & \cdots & x_{k+m-1} & x_{k+m} & x_{k+m+1} & \dots & x_r \\ x_1 & x_2 & \cdots & x_{k-1} & x_{k+1} & x_{k+2} & \cdots & x_{k+m} & ax_k & x_{k+m+1} & \dots & x_r \end{pmatrix},$$

where $k \geq 1, m \geq 0, a \in G$, lie in P.

Lemma Let $\varepsilon \neq \phi = \mathbf{p}_{\lambda i}$ where $\lambda = (u_1, \cdots, u_r)$ and $i \in I$. Then (i, λ) is connected to (j, μ) where

$$\mu = (1, \cdots, k-1, k+1, \cdots, r+1) \text{ and } j \in I.$$

Lemma Let $\mathbf{p}_{\lambda i} = \mathbf{p}_{\nu k}$ have simple form. Then $f_{i,\lambda} = f_{k,\nu}$.

Our aim here is to prove that for any $\alpha \in H$, if $i, j \in I$ and $\lambda, \mu \in \Lambda$ with $\mathbf{p}_{\lambda i} = \mathbf{p}_{\mu j} = \alpha \in H$, then $f_{i,\lambda} = f_{j,\mu}$. This property of α is called **consistency**.

Note All elements with simple form are consistent.

The result for $r \leq n-2$

How to split an arbitrary element α in H into a product of elements with simple form?

Moreover, how this splitting match the products of generators $f_{i,\lambda}$ in $\overline{H}$.

Definition Let $\alpha \in H$. We say that α has *rising point* $r+1$ if $x_m\alpha = ax_r$ for some $m \in [1, r]$ and $a \neq 1_G$; otherwise, the rising point is $k \leq r$ if there exists a sequence

$$1 \leq i < j_1 < j_2 < \cdots < j_{r-k} \leq r$$

with

$$x_i\alpha = x_k, x_{j_1}\alpha = x_{k+1}, x_{j_2}\alpha = x_{k+2}, \cdots, x_{j_{r-k}}\alpha = x_r$$

and such that if $l \in [1, r]$ with $x_l\alpha = ax_{k-1}$, then if $l < i$ we must have $a \neq 1_G$.

The result for $r \leq n-2$

Fact The only element with rising point 1 is the identity of H, and elements with rising point 2 have either of the following two forms:

(i) $\alpha = \begin{pmatrix} x_1 & x_2 & \cdots & x_r \\ ax_1 & x_2 & \cdots & x_r \end{pmatrix}$, where $a \neq 1_G$;

(ii) $\alpha = \begin{pmatrix} x_1 & x_2 & \cdots & x_{k-1} & x_k & x_{k+1} & \cdots & x_r \\ x_2 & x_3 & \cdots & x_k & ax_1 & x_{k+1} & \cdots & x_r \end{pmatrix}$, where $k \geq 2$.

Note Both of the above two forms are the so called simple forms; however, elements with simple form can certainly have rising point greater than 2, indeed, it can be $r+1$.

Lemma Let $\alpha \in H$ have rising point 1 or 2. Then α is consistent.

Lemma Every $\alpha \in P$ is consistent. Further, if $\alpha = \mathbf{p}_{\lambda j}$ then

$$f_{j,\lambda} = f_{i_1,\lambda_1} \cdots f_{i_k,\lambda_k},$$

where $\mathbf{p}_{\lambda_t,i_t}$ is an element with simple form, $t \in [1,k]$.

Idea. We proceed by induction on rising points. For any $\alpha \in H$ with rising point $k \geq 3$, we have

$$\alpha = \beta\gamma$$

for some $\beta \in H$ with rising point no more than $k-1$ and some $\gamma \in H$ with simple form. Further, this splitting matches the products of corresponding generators in $\overline{H}$.

We may denote all generators $f_{i,\lambda}$ with $\mathbf{p}_{\lambda i} = \alpha$ by f_α, where $(i,\lambda) \in K$.

The result for $r \leq n-2$

Our eventual aim is to show

$$\overline{H} \cong H \cong G \wr \mathcal{S}_r.$$

Definition We say that for $\phi, \varphi, \psi, \sigma \in P$ the quadruple $(\phi, \varphi, \psi, \sigma)$ is **singular** if $\phi^{-1}\psi = \varphi^{-1}\sigma$ and we can find $i, j \in I, \lambda, \mu \in \Lambda$ with $\phi = \mathbf{p}_{\lambda i}, \varphi = \mathbf{p}_{\mu i}, \psi = \mathbf{p}_{\lambda j}$ and $\sigma = \mathbf{p}_{\mu j}$.

Proposition Let $\overline{\overline{H}}$ be the group given by the presentation $\mathcal{Q} = \langle S : \Gamma \rangle$ with generators:

$$S = \{f_\phi : \ \phi \in P\}$$

and with the defining relations Γ :
$(P1)$ $f_\phi^{-1} f_\varphi = f_\psi^{-1} f_\sigma$ where $(\phi, \varphi, \psi, \sigma)$ is singular;
$(P2)$ $f_\epsilon = 1$.
Then $\overline{\overline{H}}$ is isomorphic to $\overline{H}$.

Put

$$\iota_{a,i} = \begin{pmatrix} x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & \cdots & x_r \\ x_1 & \cdots & x_{i-1} & ax_i & x_{i+1} & \cdots & x_r \end{pmatrix};$$

for $1 \leq k \leq r-1$.

Put

$$(k\; k+1 \cdots k+m) = \begin{pmatrix} x_1 & \cdots & x_{k-1} & x_k & \cdots & x_{k+m-1} & x_{k+m} & x_{k+m+1} & \cdots & x_r \\ x_1 & \cdots & x_{k-1} & x_{k+1} & \cdots & x_{k+m} & x_k & x_{k+m+1} & \cdots & x_r \end{pmatrix}$$

and we denote $(k\;\; k+1)$ by τ_k.

The result for $r \leq n-2$

The group $H \cong G \wr \mathcal{S}_r$ has a presentation $\mathcal{U} = \langle Y : \Upsilon \rangle$, with generators

$$Y = \{\tau_i, \iota_{a,j} : 1 \leq i \leq r-1, 1 \leq j \leq r, a \in G\}$$

and defining relations Υ:
$(W1)$ $\tau_i \tau_i = 1$, $1 \leq i \leq r-1$;
$(W2)$ $\tau_i \tau_j = \tau_j \tau_i$, $j \pm 1 \neq i \neq j$;
$(W3)$ $\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$, $1 \leq i \leq r-2$;
$(W4)$ $\iota_{a,i} \iota_{b,j} = \iota_{b,j} \iota_{a,i}$, $a, b \in G$ and $1 \leq i \neq j \leq r$;
$(W5)$ $\iota_{a,i} \iota_{b,i} = \iota_{ab,i}$, $1 \leq i \leq r$ and $a, b \in G$;
$(W6)$ $\iota_{a,i} \tau_j = \tau_j \iota_{a,i}$, $1 \leq i \neq j, j+1 \leq r$;
$(W7)$ $\iota_{a,i} \tau_i = \tau_i \iota_{a,i+1}$, $1 \leq i \leq r-1$ and $a \in G$.

Recall that

$$\overline{\overline{H}} = \langle f_\phi : \ \phi \in P \rangle,$$

and further decomposition gives

$$\overline{\overline{H}} = \langle f_{\tau_i}, f_{\iota_{a,j}} : \ 1 \leq i \leq r-1, 1 \leq j \leq r, a \in G \rangle.$$

Find a series of relations $(T1)-(T6)$ satisfied by these generators:

$(T1)$ $f_{\tau_i} f_{\tau_i} = 1$, $1 \leq i \leq r-1$.
$(T2)$ $f_{\tau_i} f_{\tau_j} = f_{\tau_j} f_{\tau_i}$, $j \pm 1 \neq i \neq j$.
$(T3)$ $f_{\tau_i} f_{\tau_{i+1}} f_{\tau_i} = f_{\tau_{i+1}} f_{\tau_i} f_{\tau_{i+1}}$, $1 \leq i \leq r-2$.
$(T4)$ $f_{\iota_{a,i}} f_{\iota_{b,j}} = f_{\iota_{b,j}} f_{\iota_{a,i}}$, $a, b \in G$ and $1 \leq i \neq j \leq r$.
$(T5)$ $f_{\iota_{b,i}} f_{\iota_{a,i}} = f_{\iota_{ab,i}}$, $1 \leq i \leq r$ and $a, b \in G$.
$(T6)$ $f_{\iota_{a,i}} f_{\tau_j} = f_{\tau_j} f_{\iota_{a,i}}$, $1 \leq i \neq j, j+1 \leq r$.
$(T7)$ $f_{\iota_{a,i}} f_{\tau_i} = f_{\tau_i} f_{\iota_{a,i+1}}$, $1 \leq i \leq r-1$ and $a \in G$.

Note A twist between (W5) and (T5).

Lemma The group $\overline{\overline{H}}$ with a presentation $\mathcal{Q} = \langle S : \Gamma \rangle$ is isomorphic to the presentation $\mathcal{U} = \langle Y : \Upsilon \rangle$ of H, so that $\overline{H} \cong H$.

The result for $r \leq n-2$

Theorem Let End $F_n(G)$ be the endomorphism monoid of a free G-act $F_n(G)$ on n generators, where $n \in \mathbb{N}$ and $n \geq 3$, let E be the biordered set of idempotents of End $F_n(G)$, and let IG(E) be the free idempotent generated semigroup over E.

For any idempotent $\varepsilon \in E$ with rank r, where $1 \leq r \leq n-2$, the maximal subgroup $\overline{H}$ of IG(E) containing $\overline{\varepsilon}$ is isomorphic to the maximal subgroup H of End $F_n(G)$ containing ε and hence to $G \wr \mathcal{S}_r$.

Note If $r = n$, then $\overline{H}$ is trivial; if $r = n-1$, then $\overline{H}$ is free.

If $r = 1$, then $H = G$ and so that:

Corollary Every group can be a maximal subgroup of a naturally occurring IG(E).

The result for $r \leq n-2$

If G is trivial, then End $F_n(G)$ is essentially $\mathcal{T}_n$, so we deduce the following result:

Corollary Let $n \in \mathbb{N}$ with $n \geq 3$ and let IG(E) be the free idempotent generated semigroup over the biordered set E of idempotents of $\mathcal{T}_n$.

For any idempotent $\varepsilon \in E$ with rank r, where $1 \leq r \leq n-2$, the maximal subgroup $\overline{H}$ of IG(E) containing $\overline{\varepsilon}$ is isomorphic to the maximal subgroup H of $\mathcal{T}_n$ containing ε, and hence to $\mathcal{S}_r$.