Left Restriction Semigroups from Incomplete Automata

Yanhui Wang

Shandong University of Science and Technology, China

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Left Restriction Semigroups

Let $S = (S, \cdot, ^+)$ be a semigroup equipped with a unary operation $^+$.

Definition S is **left restriction** if the following identities hold:

$$x^+x = x,\ x^+y^+ = y^+x^+, (x^+y)^+ = x^+y^+, xy^+ = (xy)^+x.$$

Inverse Systems

An **inverse system** of algebras and homomorphisms is $\{(A_i)_{i\in I}; f_{ji}, i \leq j\}$, where

1. $(I, \leq)$ is a directed poset, that is, for any $i, j \in I$, there exists $k \in I$ such that $i \leq k$ and $j \leq k$;
2. $(A_i)_{i\in I}$ is a family of algebras;
3. $f_{ji} : A_j \to A_i$ for all $i \leq j$ in I is a family of homomorphisms satisfying

   (1) $f_{ii} = \mathrm{Id}_{A_i}$ for all $i \in I$;
   (2) for any $i \leq j \leq k$, we have $f_{ki} = f_{ji} \circ f_{kj}$.

The **inverse limit** of an inverse system of algebras and homomorphisms $\{(A_i)_{i\in I}; f_{ji}, i \leq j\}$ is a subalgebra of $\prod_{i\in I} A_i$ defined by

$$\varprojlim_{i\in I} A_i = \{(a_i)_{i\in I} \in \prod_{i\in I} A_i | a_i = a_j f_{ji} \text{ for all } i \leq j \text{ in } I\}.$$

Wreath Products

Let X be a non-empty set. We will denote the (partial) transformation semigroup S over X by (S, X).

Let (S, X) and (T, Y) be (partial) transformation semigroups. We put

$$S \wr T = \{(g, h) : g \in S, \;\; h : \mathrm{dom}(g) \to T\}.$$

For any $(x, y) \in X \times Y$, we define

$$(x, y)^{(g,h)} = \begin{cases} (x^g, y^{xh}) & \text{if } \; x \in \; \mathrm{dom}(g), y \in \; \mathrm{dom}(xh) \\ \textit{undefined}, & \text{otherwise.} \end{cases}$$

For any (g_1, h_1), $(g_2, h_2) \in S \wr T$, and $x \in \mathrm{dom}(g_1 g_2)$,

$$(g_1, h_1)(g_2, h_2) = (g_1 g_2, h),$$

where $xh = (xh_1)(x^{g_1})h_2$.

Then $S \wr T$ forms a semigroup of (partial) transformations over $X \times Y$, called the **wreath product** of (S, X) and (T, Y).

Infinitely Iterated Wreath Products

Let (S_i, X_i) $(i \geq 1)$ be a sequence of (partial) transformation semigroups.

For arbitrary $n \geq 2$, we have a wreath product

$$W_n = \wr_{i=1}^{n}(S_i, X_i),$$

which acts by (partial) transformation on $X^{(n)} = X_1 \times X_2 \times \cdots \times X_n$, and is called the **iterated wreath product** of (S_i, X_i) $(i = 1, 2, \cdots, n)$.

For any $n \leq m$, we define a map $\phi_{m,n} : W_m \to W_n$ by the rule that for any $(s_1, s_2, \ldots, s_n, \ldots, s_m) \in W_m$,

$$(s_1, s_2, \ldots, s_n, \ldots, s_m)\phi_{m,n} = (s_1, s_2, \ldots, s_n).$$

The inverse limit of $((W_n)_{n\in\mathbb{N}}, \phi_{m,n}, n \leq m)$ is

$$\varprojlim_{n\in\mathbb{N}} W_n = \{(w_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} W_n | w_n = w_m\phi_{m,n} \text{ for all } n \leq m \text{ in } \mathbb{N}\},$$

called the **infinitely iterated wreath product** of (S_i, X_i) $(i \geq 1)$.

(Incomplete) Automata

An **(incomplete) automaton** $\mathcal{A}$ is a quadruple (Q, X, τ, λ), where

1. Q is a finite (resp. infinite) set of states
2. X is a finite alphabet
3. τ is a (partial) function from $Q \times X$ to Q
4. λ is a (partial) function from $Q \times X$ to X
5. $\big(\mathrm{dom}(\tau) = \mathrm{dom}(\lambda)\big)$.

   Further,

6. if for each $q \in Q$, $\lambda_q : X \to X$ defined by the rule that

$$x\lambda_q = (q, x)\lambda, \quad x \in X,$$

is a (partial) permutation over X, then $\mathcal{A}$ is called an **(incomplete) permutational automaton**.

Extended Functions Based on (Incomplete) Automata

Given an (incomplete) automaton $\mathcal{A} = (Q, X, \tau, \lambda)$, τ and λ can be extended as follows:

$$\tau : Q \times X^*(\text{resp. } X^\omega) \to Q : \quad (q, \varepsilon)\tau = q, \quad (q, wx)\tau = ((q, w)\tau, x)\tau$$

$$\lambda : Q \times X^*(\text{resp. } X^\omega) \to X \cup \{\varepsilon\} : \quad (q, \varepsilon)\lambda = \varepsilon, \quad (q, wx)\lambda = ((q, w)\tau, x)\lambda$$

where $q \in Q, w \in X^*(\text{resp. } X^\omega), x \in X$.

Remark: if $(q, w)\tau$ (resp. $(q, w)\lambda$) is not defined, then τ (resp. λ) is not defined for all pairs (q, u), where w is a prefix of u.

(Incomplete) Automaton Transformations

Let $\mathcal{A} = (Q, X, \tau, \lambda)$ be an (incomplete) automaton.

For any $q \in Q$, $u = x_1x_2x_3\cdots \in X^*$ (resp. $u \in X^\omega$), we define

$$\begin{aligned} uf_{\mathcal{A},q} &= (q, x_1)\lambda(q, x_1x_2)\lambda(q, x_1x_2x_3)\lambda\cdots \\ &= (q, x_1)\lambda((q, x_1)\tau, x_2)\lambda((q, x_1x_2)\tau, x_3)\lambda\cdots . \end{aligned}$$

We call $f_{\mathcal{A},q}$ a **(partial) automaton transformation** over X^* (resp. X^ω).

In an incomplete automaton $\mathcal{A}$, $f_{\mathcal{A},q}$ is called a **partial automaton permutation** if its restriction to the domain is injective.

Groups and Semigroups from (Incomplete) Automata

1. if $\mathcal{A}$ is a permutational automaton with finite set of states Q, then $G(\mathcal{A}) = \langle f_{\mathcal{A},q} : q \in Q \cup Q^{-1} \rangle$ is a group. A group G is called an automaton group if $G \cong G(\mathcal{A})$;
2. if $\mathcal{A}$ is an automaton with finite set of states Q, then $\Sigma(\mathcal{A}) = \langle f_{\mathcal{A},q} : q \in Q \rangle$ is a semigroup. A semigroup S is called an automaton semigroup if $S \cong \Sigma(\mathcal{A})$.

In the following, (incomplete) automata have an infinite state set.

1. $\bigcup \{ f_{\mathcal{A},q} : q \in Q \cup Q^{-1}, \mathcal{A}$ is an permutational automaton over $X\}$ forms a group $GA(X)$;
2. $\bigcup \{ f_{\mathcal{A},q} : q \in Q, \mathcal{A}$ is an automaton over $X\}$ forms a monoid $AS(X)$;
3. $\bigcup \{ f_{\mathcal{A},q} : q \in Q, \mathcal{A}$ is an incomplete permutational automaton over $X\}$ forms an inverse semigroup $ISA(X)$;
4. $\bigcup \{ f_{\mathcal{A},q} : q \in Q, \mathcal{A}$ is an incomplete automaton over $X\}$ forms a left restriction semigroup $PAS(X)$.

$PAS(X)$

Lemma 1 $PAS(X)$ is a subsemigroup of $\mathcal{PT}(X^*)$.

Lemma 2 All partial automaton identities over X form a semilattice. We denote it by $EA(X^*)$.

Let $\mathcal{A} = (Q, X, \tau, \lambda)$ be an incomplete automaton.
We define $\mathcal{A}^+ = (Q, X, \tau^+, \lambda^+)$, where for any $q \in Q$ and $x \in X$,

$$(q, x)\tau^+ = \begin{cases} p & \text{if } (q, x)\tau = p \\ \textit{undefined}, & \text{otherwise.} \end{cases}$$

and

$$(q, x)\lambda^+ = \begin{cases} x & \text{if } (q, x) \in \operatorname{dom}(\lambda) \\ \textit{undefined}, & \text{otherwise.} \end{cases}$$

For any $q \in Q$, we have $f^+_{\mathcal{A},q} = f_{\mathcal{A}^+,q}$.

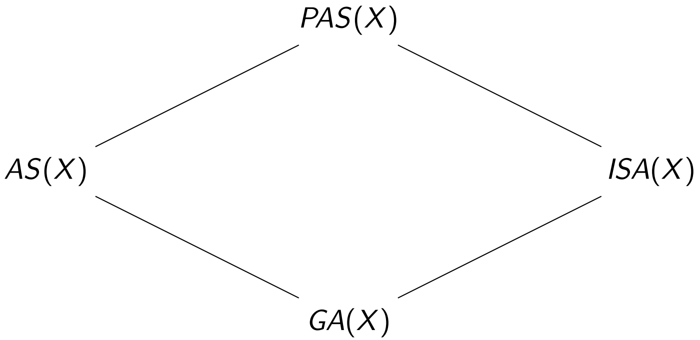
$PAS(X)$

Remark:

1. for any $f_{\mathcal{A},q} \in PAS(X)$ and $l_{\mathcal{B},q'} \in EA(X^*)$, we have

$$(f_{\mathcal{A},q} l_{\mathcal{B},q'})^+ f_{\mathcal{A},q} = f_{\mathcal{A},q} l_{\mathcal{B},q'}$$

Lemma 3 $PAS(X)$ forms a left restriction semigroup.



Results

Let X be a finite alphabet such that $|X| \geq 2$.

1. The group $GA(X)$ is isomorphic to the infinitely iterated wreath product of the symmetric group $Sym(X)$ of X.
2. The monoid $AS(X)$ is isomorphic to the infinitely iterated wreath product of the transformation monoid $\mathcal{T}(X)$ of X.
3. The inverse semigroup $ISA(X)$ is isomorphic to the infinitely iterated wreath product of the symmetric inverse semigroup $\mathcal{IS}(X)$ of X.

The Infinitely Iterated Wreath Product of $\mathcal{PT}(X)$

A construction:

1. $PT(X)^n = \wr_{i=1}^{n} \mathcal{PT}(X)$, for any $n \in \mathbb{N}$.
2. The collection $((PT(X)^n)_{n\in\mathbb{N}}, \phi_{m,n}, n \leq m)$ is an inverse system.
3. $\varprojlim_{n\in\mathbb{N}} PT(X)^n$
   $= \{(w_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} PT(X)^n | w_n = w_m\phi_{m,n}$ for all $n \leq m$ in $\mathbb{N}\}$.

Remark:

1. The element of $\varprojlim_{n\in\mathbb{N}} PT(X)^n$ is in the form

$$((\sigma_1), (\sigma_1, \sigma_2), (\sigma_1, \sigma_2, \sigma_3), \cdots, (\sigma_1, \sigma_2, \cdots, \sigma_n), \cdots),$$

   where $(\sigma_1, \sigma_2, \cdots, \sigma_n) \in PT(X)^n$.

2. $\{((x_1), (x_1, x_2), \cdots, (x_1, x_2, \cdots, x_n), \cdots) | x_i \in X_i, i \geq 1\} = \prod_{i\geq 1} X_i$

$\varprojlim_{n\in\mathbb{N}} PT(X)^n \to PAS(X)$

For any element s of $\varprojlim_{n\in\mathbb{N}} PT(X)^n$ is in the form

$$s = [s_1, s_2, \cdots, s_n, \cdots]$$

where $s_1 \in \mathcal{PT}(X)$ and $s_n : X^{n-1} \to \mathcal{PT}(X)$ for each $n \geq 2$.

Remark:

1. $x = (x_n)_{n\geq 1} \in X^\omega$ is contained in dom(s) if and only if $x_1 \in$ dom(s_1) and $x_n \in$ dom$((x_1x_2\cdots x_{n-1})s_n)$ for $n \geq 2$.
2. $x^s = (x_1^{s_1}, x_2^{(x_1)s_2}, \cdots, x_n^{(x_1x_2\cdots x_{n-1})s_n}, \cdots)$

Lemma 4 A partial transformation $f : X^\omega \to X^\omega$ is a partial automaton transformation if and only if it preserves the lengths of common beginnings of ω-words.

$s = [s_1, s_2, \cdots, s_n, \cdots] \in PAS(X)$.

$PAS(X) \to \varprojlim_{n\in\mathbb{N}} PT(X)^n$

Let $f \in PAS(X)$. Then there exists an incomplete automaton $\mathcal{A} = \{Q, X, \tau, \lambda\}$ and $q \in Q$ such that $f = f_{\mathcal{A},q}$.

Notice:

For any $u = x_1x_2x_3\cdots \in X^\omega$, we have

$$\begin{aligned} uf_{\mathcal{A},q} &= (q, x_1)\lambda(q, x_1x_2)\lambda(q, x_1x_2x_3)\lambda\cdots. \\ &= (q, x_1)\lambda((q, x_1)\tau, x_2)\lambda((q, x_1x_2)\tau, x_3)\lambda\cdots. \end{aligned}$$

We define

$$f_1 = \lambda_q \in \mathcal{PT}(X)$$

$$f_n : X^{n-1} \to \mathcal{PT}(X) \text{ by } (x_1x_2\cdots x_{n-1})f_n = \lambda_{(q,x_1x_2\cdots x_{n-1})\tau}$$

Then $f = [f_1, f_2, \cdots, f_n, \cdots] \in \varprojlim_{n\in\mathbb{N}} PT(X)^n$.

The Main Result

Let X be a finite alphabet such that $|X| \geq 2$.

Theorem The left restriction semigroup $PAS(X)$ is isomorphic to the infinitely iterated wreath product of the partial transformation semigroup $\mathcal{PT}(X)$ of X.