On congruence extension property for ordered algebras

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Definitions

The *type* of an algebra is a (possibly empty) set Ω which is a disjoint union of sets Ω_k, $k \in \mathbb{N} \cup \{0\}$.

Definition 1 Let Ω be a type. An *ordered Ω-algebra* is a triplet $\mathcal{A} = (A, \Omega_A, \leq_A)$ comprising a poset $(A, \leq_A)$ and a set Ω_A of operations on A (for every k-ary operation symbol $\omega \in \Omega_k$ there is a k-ary operation $\omega_A \in \Omega_A$ on A) such that all the operations ω_A are monotone mappings, where monotonicity of ω_A $(\omega \in \Omega_k)$ means that

$$a_1 \leq_A a_1' \wedge \ldots \wedge a_k \leq_A a_k' \Longrightarrow \omega_A(a_1, \ldots, a_k) \leq_A \omega_A(a_1', \ldots, a_k')$$

for all $a_1, \ldots, a_k, a_1', \ldots, a_k' \in A$.

A *homomorphism* $f : \mathcal{A} \longrightarrow \mathcal{B}$ of ordered algebras is a monotone operation-preserving map from an ordered Ω-algebra $\mathcal{A}$ to an ordered Ω-algebra $\mathcal{B}$. A *subalgebra* of an ordered algebra $\mathcal{A} = (A, \Omega_A, \leq_A)$ is a subset B of A, which is closed under operations and equipped with the order $\leq_B \; = \; \leq_A \cap (B \times B)$. On the direct product of ordered algebras the order is defined componentwise.

Definition 2 A class of ordered Ω-algebras is called a *variety*, if it is closed under isomorphisms, quotients, subalgebras and products.

Every variety of ordered Ω-algebras together with their homomorphisms forms a category.

An *inequality* of type Ω is a sequence of symbols $t \leq t'$, where t, t' are Ω-terms. We say that "$t \leq t'$ holds in an ordered algebra $\mathcal{A}$" if $t_{\mathcal{A}} \leq t'_{\mathcal{A}}$ where $t_{\mathcal{A}}, t'_{\mathcal{A}} : A^n \to A$ are the functions on $\mathcal{A}$ induced by t and t'. Inequalities $t \leq t'$ and $t' \leq t$ hold if and only if the identity $t = t'$ holds. A class $\mathcal{K}$ of Ω-algebras is a variety if and only if it consists precisely of all the algebras satisfying some set of inequalities.

Example 1 Lattices, bounded posets, posemigroups or pomonoids form a variety.

If S is a pomonoid then the class of all right S-posets is a variety of ordered Ω-algebras, where $\Omega = \Omega_1 = \{\cdot s \mid s \in S\}$, defined by the following set of identities and inequalities:

$$\{(x \cdot s) \cdot t = x \cdot (st) \mid s, t \in S\} \cup \{x \cdot 1 = x\} \cup \{x \cdot s \leq x \cdot t \mid s, t \in S, s \leq t\}.$$

If θ is a preorder on a poset $(A, \leq)$ and $a, a' \in A$ then we write

$$a \underset{\theta}{\leq} a' \Longleftrightarrow (\exists n \in \mathbb{N})(\exists a_1, \ldots, a_n \in A)(a \leq a_1 \theta a_2 \leq a_3 \theta \ldots \theta a_n \leq a').$$

Definition 3 *An* **order-congruence** *on an ordered algebra $\mathcal{A}$ is an algebraic congruence θ such that the following condition is satisfied,*

$$(\forall a, a' \in A)\left(a \underset{\theta}{\leq} a' \underset{\theta}{\leq} a \Longrightarrow a\theta a'\right).$$

We call a preorder σ on an ordered algebra $\mathcal{A}$ a **lax congruence** *if it is compatible with operations and extends the order of $\mathcal{A}$.*

CEP and LEP

Definition 4 We say that an ordered algebra $\mathcal{A}$ has the **congruence extension property** (CEP for short) if every order-congruence θ on an arbitrary subalgebra $\mathcal{B}$ of $\mathcal{A}$ is induced by an order-congruence Θ on $\mathcal{A}$, i.e. $\Theta \cap (B \times B) = \theta$.

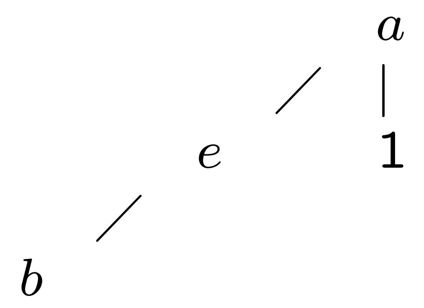
Definition 5 We say that an ordered algebra $\mathcal{A}$ has the **lax congruence extension property** (LEP for short) if every lax congruence σ on an arbitrary subalgebra $\mathcal{B}$ of $\mathcal{A}$ is induced by a lax congruence Σ on $\mathcal{A}$, i.e. $\Sigma \cap (B \times B) = \sigma$.

Proposition 1 *If an ordered algebra has LEP then it has CEP.*

Example 2 Consider a pomonoid S with the following multiplication table and order:

$\cdot$	a	b	e	1
a	a	a	a	a
b	b	b	b	b
e	a	b	e	e
1	a	b	e	1

,



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Then S has CEP. To show that S does not have LEP we consider a lax congruence

$$\theta = \{(b,e),(e,b),(b,b),(e,e),(1,1)\}$$

on a subpomonoid $U = \{b,e,1\}$. Now any lax congruence Γ on S extending θ must contain $(a,b) = (ea,ba)$. Since Γ extends the order of S, it shoud also contain the pair $(1,a)$. Using the transitivity we have $(1,b) \in \Gamma$ from $(1,a),(a,b) \in \Gamma$. Therefore $\Gamma \cap (U \times U) \neq \theta$ because $(1,b) \notin \theta$, and LEP fails for S.

Proposition 2 *Let* $\Omega = \Omega_0 \cup \Omega_1$. *Then every ordered* Ω*-algebra has LEP (and hence CEP).*

Corollary 1 *Every poset has LEP.*

Corollary 2 *If* S *is a posemigroup or a pomonoid then every* S*-poset has LEP.*

LEP and diagrams

Definition 6 A mapping $f : \mathcal{A} \to \mathcal{B}$ between posets $\mathcal{A} = (A, \leq_A)$ and $\mathcal{B} = (B, \leq_B)$ is called an **order embedding** if

$$a \leq_A a' \Longleftrightarrow f(a) \leq_B f(a')$$

for all $a, a' \in A$.

Proposition 3 *An algebra $\mathcal{A}$ in a variety $\mathcal{V}$ of ordered Ω-algebras has LEP if and only if for each order embedding $f : \mathcal{B} \longrightarrow \mathcal{A}$ and surjective morphism $g : \mathcal{B} \longrightarrow \mathcal{C}$ there exist an order embedding $f' : \mathcal{C} \longrightarrow \mathcal{D}$ and a homomorphism $g' : \mathcal{A} \longrightarrow \mathcal{D}$ such that $g'f = f'g$.*

Shortly:

$$
LEP \iff \begin{array}{ccc}
\mathcal{B} & \xrightarrow{f \text{ ord. emb.}} & \mathcal{A} \\
{\scriptstyle g \text{ surj.}}\downarrow & & \downarrow{\scriptstyle g'} \\
\mathcal{C} & \underset{f' \text{ ord. emb.}}{\dashrightarrow} & \mathcal{D}
\end{array}
$$

Definition 7 A morphism $g : \mathcal{B} \to \mathcal{C}$ of ordered Ω-algebras is a **regular epimorphism** if

$$(\forall c, c' \in \mathcal{C})(\exists b, b' \in \mathcal{B})(c = g(b) \land c' = g(b') \land b \underset{\ker g}{\leq} b').$$

Proposition 4 *An algebra $\mathcal{A}$ in a variety $\mathcal{V}$ of ordered Ω-algebras has CEP if and only if for each injective homomorphism $f : \mathcal{B} \longrightarrow \mathcal{A}$ and regular epimorphism $g : \mathcal{B} \longrightarrow \mathcal{C}$ there exist an injective homomorphism $f' : \mathcal{C} \longrightarrow \mathcal{D}$ and a homomorphism $g' : \mathcal{A} \longrightarrow \mathcal{D}$ such that $g'f = f'g$.*

Shortly:

$$\text{CEP} \Longleftrightarrow \quad \begin{array}{ccc} \mathcal{B} & \xrightarrow{f \text{ inj.}} & \mathcal{A} \\ {\scriptstyle g \text{ reg. epi}} \downarrow & & \downarrow {\scriptstyle g'} \\ \mathcal{C} & \underset{f' \text{ inj.}}{\dashrightarrow} & \mathcal{D} \end{array} .$$

Definition 8 We say that an unordered algebra $\mathcal{A}$ has the **strong congruence extension property (SCEP)** if any order-congruence θ on a subalgebra $\mathcal{B}$ of $\mathcal{A}$ can be extended to an order-congruence Θ on $\mathcal{A}$ in such a way that $\Theta \cap (B \times A) = \theta$.

Proposition 5 *An algebra $\mathcal{A}$ in a variety $\mathcal{V}$ of ordered Ω-algebras has SCEP if and only if each diagram*

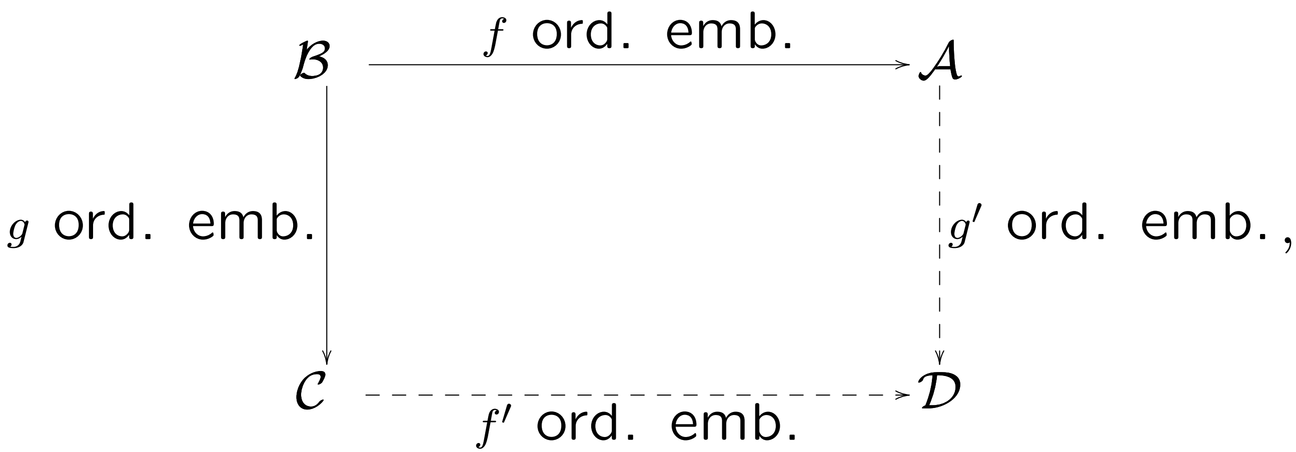
$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\ f\ } & \mathcal{A} \\ {\scriptstyle g}\big\downarrow & & \\ \mathcal{C} & & \end{array}$$

where f is an order embedding and g is a regular epimorphism, *can be completed to a pullback diagram*

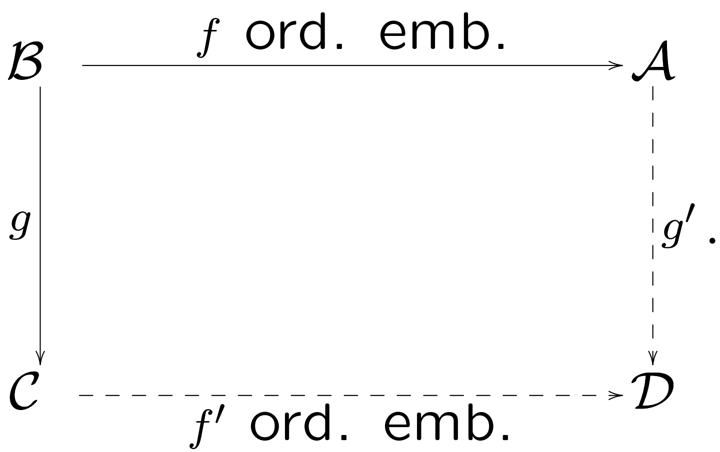
$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{f} & \mathcal{A} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ \mathcal{C} & \xrightarrow[f']{} & \mathcal{D} \end{array}$$

in $\mathcal{V}$, where f' is an injective homomorphism.

Amalgamation property (AP):



transferability property (TP):



Proposition 6 *In a class $\mathcal{K}$ of ordered Ω-algebras we have the following.*

1. *If $\mathcal{K}$ is closed under quotients then LEP and AP imply TP.*

2. *TP implies LEP.*

3. *If $\mathcal{K}$ has finite products then $\mathcal{K}$ has TP iff it has LEP and AP.*

The case of Hamiltonian algebras

An unordered algebra $\mathcal{A}$ is called **Hamiltonian** if every subalgebra $\mathcal{B}$ of $\mathcal{A}$ is a class of a suitable congruence on $\mathcal{A}$. A variety is called **Hamiltonian** if all its algebras are Hamiltonian.

An unordered algebra is said to have the **strong congruence extension property** (SCEP) if any congruence θ on a subalgebra $\mathcal{B}$ of an algebra $\mathcal{A}$ can be extended to a congruence Θ of $\mathcal{A}$ in such a way that each Θ-class is either contained in B or disjoint from B. The last means that $\Theta \cap (B \times A) = \theta$.

Theorem 1 (Kiss; Gould and Wild) *If $\mathcal{A}$ is an algebra such that $\mathcal{A} \times \mathcal{A}$ is Hamiltonian, then $\mathcal{A}$ has the SCEP. In particular, each Hamiltonian variety of unordered algebras has the SCEP.*

If θ is an order-congruence on an ordered algebra $\mathcal{A}$ then every θ-class is a convex subset of $\mathcal{A}$.

We say that an ordered algebra $\mathcal{A}$ is **Hamiltonian** if every convex subalgebra $\mathcal{B}$ of $\mathcal{A}$ is a class of a suitable order-congruence on $\mathcal{A}$.

Example 3 The variety of S-posets is Hamiltonian, because of the Rees congruences.

Proposition 7 *Let $\mathcal{B}$ be an up-closed subalgebra of an ordered algebra $\mathcal{A}$, where $\mathcal{A} \times \mathcal{A}$ is Hamiltonian. If σ is a lax congruence on $\mathcal{B}$ which is a convex subset of $B \times B$ then σ can be extended to a lax congruence Σ on $\mathcal{A}$ in such a way that $\Sigma \cap (B \times A) = \sigma$.*

Proposition 8 *Let $\mathcal{B}$ be an up-closed subalgebra of an ordered algebra $\mathcal{A}$, where $\mathcal{A}\times\mathcal{A}$ is Hamiltonian. If θ is an order-congruence on $\mathcal{B}$ which extends the order of $\mathcal{B}$ then θ can be extended to an order-congruence on $\mathcal{A}$.*

Proposition 9 *Let $\mathcal{B}$ be an up-closed (or down-closed) subalgebra of an ordered algebra $\mathcal{A}$. Assume that $\mathcal{A}$ has the algebraic SCEP with respect to $\mathcal{B}$. Then every order-congruence of $\mathcal{B}$ can be extended to $\mathcal{A}$.*

References

1. Biró, B.; Kiss, E.W.; Pálfy, P.P. On the congruence extension property. Universal algebra (Esztergom, 1977), pp. 129151, Colloq. Math. Soc. János Bolyai, 29, North-Holland, Amsterdam-New York, 1982.

2. Bloom S.L., *Varieties of ordered algebras*, J. Comput. System Sci. **13** (1976), 200–212.

3. Czédli G., Lenkehegyi A., *On congruence n-distributivity of ordered algebras*, Acta Math. Hungar. **41** (1983), 17–26.

4. Czédli G., Lenkehegyi A., *On classes of ordered algebras and quasiorder distributivity*, Acta Sci. Math. **46** (1983), 41–54.

5. Gould, V., Wild, M., *Every Hamiltonian variety has the congruence extension property — a short proof*, Algebra Universalis **26** (1989), 187–188.

6. Kiss, E.W., *Each Hamiltonian variety has the congruence extension property*, Algebra Universalis **12** (1981), 395398.

7. Kiss E. W., Márki L., Pröhle P., Tholen W., *Categorical algebraic properties. A compendium on amalgamation, congruence extension, epimorphisms, residual smallness, and injectivity*, Studia Sci. Math Hung. **18** (1983), 79-141.