Endomorphisms of semigroups: Growth and interactions with subsemigroups

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(Joint work with Victor Maltcev)

Endomorphism growth

- S a finitely generated semigroup.
- T a subsemigroup of S.
- A a finite generating set for S.
- $|x|_A$ the length of $x \in S$ with respect to A; that is, the minimum length of a product of elements of A that equals x.
- ϕ endomorphism of S.

For $m \in \mathbb{N}$, the growth function of ϕ with respect to elements of length m over A is defined by

$$\Gamma_{\phi,m,A}(n) = \max\left\{\left|w\phi^n\right| : |w|_A \leqslant m\right\}.$$

The growth of ϕ is defined by

$$\Gamma(\phi) = \sup\left\{\limsup_{n\to\infty} \sqrt[n]{\Gamma_{\phi,m,A}(n)} : m \in \mathbb{N}\right\}.$$

Endomorphism growth

$$\Gamma_{\phi,m,A}(n) = \max\{\left|w\phi^n\right| : |w|_A \leqslant m\}$$

$$\Gamma(\phi) = \sup\{\limsup_{n\to\infty} \sqrt[n]{\Gamma_{\phi,m,A}(n)} : m \in \mathbb{N}\}$$

Example

Let $S = \{a\}^+$, $A = \{a\}$, and $a\phi = a^2$. Then

$$\Gamma_{\phi,m,A}(n) = \left|a^m\phi^n\right| = \left|a^{2^n m}\right| = 2^n m,$$

and so

$$\Gamma(\phi) = \sup\{\limsup_{n\to\infty} \sqrt[n]{2^n m} : m \in \mathbb{N}\} = 2.$$

Properties of endomorphism growth

$$\Gamma_{\phi,m,A}(n) = \max\left\{\left|w\phi^n\right| : |w|_A \leqslant m\right\}$$

$$\Gamma(\phi) = \sup\left\{\limsup_{n\to\infty} \sqrt[n]{\Gamma_{\phi,m,A}(n)} : m \in \mathbb{N}\right\}$$

Proposition

$\Gamma(\phi) = \lim_{n\to\infty} \sqrt[n]{\Gamma_{\phi,1,A}(n)} = \inf\{\sqrt[n]{\Gamma_{\phi,1,A}(n)} : n \in \mathbb{N}\}$.

Proof.

First, $|(a_1 \cdots a_m)\phi^n| \leqslant |a_1\phi^n| \cdots |a_m\phi^n|$;

and thus $\Gamma_{\phi,m,A}(n) \leqslant m\Gamma_{\phi,1,A}(n)$.

Also,

$$\begin{aligned} &\Gamma_{\phi,1,A}(n) \\ = {}& \max\{|a\phi^n| : a \in A\} \\ = {}& \max\{|(w_1 \cdots w_k)\phi| : w_1 \cdots w_k = a\phi^{n-1}, a \in A\} \\ \leqslant {}& \max\{|w_1\phi| \cdots |w_k\phi| : w_1 \cdots w_k = a\phi^{n-1}, a \in A\} \\ \leqslant {}& \Gamma_{\phi,1,A}(1)\Gamma_{\phi,1,A}(n-1). \end{aligned}$$

Properties of endomorphism growth

$$\Gamma_{\phi,m,A}(n) = \max\{ \left| w\phi^n \right| : |w|_A \leqslant m \}$$

$$\Gamma(\phi) = \sup\{ \limsup_{n\to\infty} \sqrt[n]{\Gamma_{\phi,m,A}(n)} : m \in \mathbb{N} \}$$

Proposition

$\Gamma(f) = \lim_{n\to\infty} \sqrt[n]{\Gamma_{f,1,A}(n)} = \inf\{ \sqrt[n]{\Gamma_{f,1,A}(n)} : n \in \mathbb{N}\}$.

Proof (continued).

So far, we have $\Gamma_{\phi,m,A}(n) \leqslant m\Gamma_{\phi,1,A}(n)$ and $\Gamma_{\phi,1,A}(n) \leqslant \Gamma_{\phi,1,A}(1)\Gamma_{\phi,1,A}(n-1)$.

Hence $\quad \Gamma(\phi) \leqslant \sup\{ \limsup_{n\to\infty} \sqrt[n]{m\Gamma_{\phi,1,A}(n)} : m \in \mathbb{N} \}$

$$\Gamma(\phi) = \limsup_{n\to\infty} \sqrt[n]{\Gamma_{\phi,1,A}(n)}.$$

Furthermore, $(\Gamma_{\phi,A}(n))^{1/n}$ is non-increasing in n, so

$$\Gamma(\phi) = \lim_{n\to\infty} \sqrt[n]{\Gamma_{\phi,1,A}(n)} = \inf\{ \sqrt[n]{\Gamma_{\phi,1,A}(n)} : n \in \mathbb{N} \}. \quad \square$$

Attainable growths

Theorem (C, Maltcev)

For any real number $r \geqslant 1$, there exists an endomorphism whose growth is r.

Proof.

Growth of the identity endomorphism is 1, so consider $r > 1$.
Let $p_n = \lceil r^{n+1} \rceil + n$ for all $n \in \mathbb{N} \cup \{0\}$. Note that $2 \leqslant p_0 < p_1 < p_2 < \cdots$.
Define the semigroup S by the following rewriting system over $A = \{a, b\}$:

$$a^{p_h}(a^{p_i}b^{p_i}ab)^{p_h}a(a^{p_i}b^{p_i}ab) \to a^{p_{i+h+1}}b^{p_{i+h+1}}ab \qquad \text{for } i, h \in \mathbb{N} \cup \{0\}.$$

This rewriting system is

- confluent, since there are no non-trivial overlaps between left-hand sides;
- noetherian, since it is length-reducing.

Attainable growths

$$a^{p_h}(a^{p_i}b^{p_i}ab)^{p_h}a(a^{p_i}b^{p_i}ab) \to a^{p_{i+h+1}}b^{p_{i+h+1}}ab$$

Proof (continued).

Define ϕ by $a \mapsto a$ and $b \mapsto a^{p_0}b^{p_0}ab$. Then ϕ is a well-defined endomorphism, since

$$\begin{aligned}
&\big(a^{p_h}(a^{p_i}b^{p_i}ab)^{p_h}a(a^{p_i}b^{p_i}ab)\big)\phi \\
&= a^{p_h}\big(\underline{a^{p_i}(a^{p_0}b^{p_0}ab)^{p_i}a(a^{p_0}b^{p_0}ab)}\big)^{p_h} \\
&\qquad a\big(\underline{a^{p_i}(a^{p_0}b^{p_0}ab)^{p_i}a(a^{p_0}b^{p_0}ab)}\big) \\
&\to \underline{a^{p_h}(a^{p_{i+1}}b^{p_{i+1}}ab)^{p_h}a(a^{p_{i+1}}b^{p_{i+1}}ab)} \\
&\to a^{p_{i+h+2}}b^{p_{i+h+2}}ab
\end{aligned}$$

and

$$\begin{aligned}
(a^{p_{i+h+1}}b^{p_{i+h+1}}ab)\phi &= \underline{a^{p_{i+h+1}}\big(a^{p_0}b^{p_0}ab\big)^{p_{i+h+1}}a\big(a^{p_0}b^{p_0}ab\big)} \\
&\to a^{p_{i+h+2}}b^{p_{i+h+2}}ab.
\end{aligned}$$

Attainable growths

$$\Gamma(\phi) = \lim_{n\to\infty} \sqrt[n]{\Gamma_{\phi,1,A}(n)} = \inf\{ \sqrt[n]{\Gamma_{\phi,1,A}(n)} : n \in \mathbb{N}\}$$

$$\phi : \quad a \mapsto a, \quad b \mapsto a^{p_0} b^{p_0} ab \qquad\qquad p_n = \lceil r^{n+1} \rceil + n$$

Proof (continued).

Since ϕ fixes a, we have $|a\phi^n| = 1$.
By induction, $b\phi^n = a^{p_{n-1}} b^{p_{n-1}} ab$, so

$$\Gamma_{\phi,1,A}(n) = |b\phi^n| = 2(n - 1 + \lceil r^n \rceil) + 2 = 2n + 2\lceil r^n \rceil.$$

Hence

$$\Gamma(\phi) = \lim_{n\to\infty} \sqrt[n]{2n + 2\lceil r^n \rceil} = r.$$

□

Homogeneous semigroup presentations

- A presentation $\langle A \mid \mathcal{R} \rangle$ is homogeneous if for all $(u, v) \in \mathcal{R}$, $|u| = |v|$.
- So if $w, w' \in A^+$ are equal in the semigroup, $|w| = |w'|$.

Let $x_{ij}^{(n)}$ the number of letters a_j in $a_i\phi^n$.

Let P be the matrix $[x_{ij}^{(1)}]$. Then

$$\begin{bmatrix} x_{i1}^{(n+1)} \\ \vdots \\ x_{ik}^{(n+1)} \end{bmatrix} = \begin{bmatrix} x_{11}^{(1)} & \cdots & x_{1k}^{(1)} \\ \vdots & \ddots & \vdots \\ x_{k1}^{(1)} & \cdots & x_{kk}^{(1)} \end{bmatrix} \begin{bmatrix} x_{i1}^{(n)} \\ \vdots \\ x_{ik}^{(n)} \end{bmatrix} = P \begin{bmatrix} x_{i1}^{(n)} \\ \vdots \\ x_{ik}^{(n)} \end{bmatrix}$$

and so,

$$|a_i\phi^n| = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} P^{n-1} \begin{bmatrix} x_{1i}^{(1)} \\ \vdots \\ x_{ki}^{(1)} \end{bmatrix}.$$

Homogeneous semigroup presentations

- If $x_{i1}^{(1)} + \cdots + x_{ik}^{(1)} = 0$ for some i, then ϕ maps S onto the subsemigroup $T = \langle a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_k \rangle$, which is also a homogeneous semigroup. This reduces the calculation of $\Gamma(\phi)$ to the calculation of $\Gamma(\phi|_T)$.
- If $x_{i1}^{(1)} + \cdots + x_{ik}^{(1)} > 0$ for all i, then

$$\Gamma(\phi) = \lim_{n\to\infty} \sqrt[n]{\|P^n\|},$$

where $\|P^n\|$ is the sum of the entries of P^n.

By Gelfand's formula,

$$\Gamma(\phi) = \lim_{n\to\infty} \sqrt[n]{\|P^n\|} = \rho(P).$$

where $\rho(P)$ is the spectral radius of P.

Proposition (C, Maltcev)

The growth of an endomorphism of a homogeneous semigroup is an algebraic number.

Mapping into a subsemigroup

Proposition (C, Maltcev)

Let T be a finitely generated subsemigroup of S, and suppose $S\phi \subseteq T$. Then $\Gamma(\phi) = \Gamma(\phi|_T)$.

Proof.

Let B generate T and let $A \supset B$ generate S.

$\leqslant$ Let $a \in A$. Then $a\phi \in T$ and so

$$|a\phi^{n+1}|_A \leqslant |a\phi^{n+1}|_B = |(a\phi)\phi|_T^n|_B \leqslant |a\phi|_B \Gamma_{\phi|_T,1,B}(n).$$

Thus

$$\Gamma_{\phi,1,A}(n+1) \leqslant \max_{a\in A} |a\phi|_B \Gamma_{\phi,1,B}(n) = k\Gamma_{\phi|_T,1,B}(n)$$

and so $\Gamma(\phi) \leqslant \Gamma(\phi|_T)$.

Mapping into a subsemigroup

Proposition (C, Maltcev)

Let T be a finitely generated subsemigroup of S, and suppose $S\phi \subseteq T$. Then $\Gamma(\phi) = \Gamma(\phi|_T)$.

Proof (continued).

Let B generate T and let $A \supset B$ generate S.

- $\geqslant$ Let $b \in B$. Let $p = |b\phi^n|_A$, so that $b\phi^n = a_1 \cdots a_p$.
  Then

  $$|b\phi^{n+1}|_B = |(a_1\phi)\cdots(a_p\phi)|_B \leqslant Mp, \text{where } M = \max_{a\in A} |a\phi|_B.$$

  Thus $|b\phi^{n+1}|_B \leqslant M\Gamma_{\phi,1,A}(n)$. This implies $\Gamma_{\phi|_T,1,B}(n+1) \leqslant M\Gamma_{\phi,1,A}(n)$, and hence $\Gamma(\phi|_T) \leqslant \Gamma(\phi)$. □

General subsemigroups

Let ϕ be such that $T\phi \subseteq T$.

In general, there is no connection between the $\Gamma(\phi)$ and $\Gamma(\phi|_T)$. That is, both $\Gamma(\phi|_T) < \Gamma(\phi)$ and $\Gamma(\phi) < \Gamma(\phi_T)$ are possible.

Example

Let $S = (\{a\}^+)^0$, let $T = \{0\}$ and define ϕ by $a \mapsto a^2$ and $0 \mapsto 0$.

Then $\Gamma(\phi) = 2$ and $\Gamma(\phi|_T) = 1$. So $\Gamma(\phi|_T) < \Gamma(\phi)$.

General subsemigroups

Example (continued)

Let ϕ be the endomorphsim of $\{a, b, c, d\}^+$ defined by

$$a \mapsto ab, \quad b \mapsto ba, \quad c \mapsto c, \quad d \mapsto d.$$

Let S be defined by the following rewriting system over $\{a, b, c, d\}$:

$$\begin{aligned}
a^n c^n a^n d &\to a\phi^n && \text{for } n \geqslant 1;\\
b^n c^n b^n d &\to a\phi^n && \text{for } n \geqslant 1;\\
(a\phi^k)^n c^n (a\phi^k)^n d &\to a\phi^{k+n} && \text{for } k, n \geqslant 1;\\
(b\phi^k)^n c^n (b\phi^k)^n d &\to b\phi^{k+n} && \text{for } k, n \geqslant 1.
\end{aligned}$$

This system is

- noetherian, since every rule reduces the number of symbols c;
- confluent, since if two left-hand sides overlap, the exponents n must coincide and $(x\phi^k)^n = (y\phi^\ell)^n$ for $x, y \in \{a, b\}$ and $k, \ell, n \in \mathbb{N}$ if and only if $k = \ell$ and $x = y$.

General subsemigroups

Example (continued)

Let ϕ be the endomorphsim of $\{a, b, c, d\}^+$ defined by

$$a \mapsto ab, \quad b \mapsto ba, \quad c \mapsto c, \quad d \mapsto d.$$

Let S be defined by the following rewriting system over $\{a, b, c, d\}$:

$$\begin{aligned} a^n c^n a^n d &\to a\phi^n && \text{for } n \geqslant 1; \\ b^n c^n b^n d &\to a\phi^n && \text{for } n \geqslant 1; \\ (a\phi^k)^n c^n (a\phi^k)^n d &\to a\phi^{k+n} && \text{for } k, n \geqslant 1; \\ (b\phi^k)^n c^n (b\phi^k)^n d &\to b\phi^{k+n} && \text{for } k, n \geqslant 1. \end{aligned}$$

The map ϕ gives an endomorphism of S. Let $T = \langle a, b \rangle = \{a, b\}^+$.
Then $T\phi \subseteq T$, and $\Gamma(\phi|_T) = 2$.

But max$\{a\phi^n, b\phi^n, x\phi^n, d\phi^n\} \leqslant 3n + 1$ and so $\Gamma(\phi) = 1$.

Thus $\Gamma(\phi_T) < \Gamma(\phi)$.

Special types of subsemigroups

Proposition (C, Maltcev)

Suppose T is finitely generated and there is a finite set $R \subseteq S$ such that $S = RT$, and ϕ is such that $T\phi \subseteq T$. Then $\Gamma(\phi) \leqslant \Gamma(\phi|_T)$.

For $x, y \in S$, define

$$\begin{aligned} x \,\mathcal{R}^T\, y &\iff xT \cup \{x\} = yT \cup \{y\}, \\ x \,\mathcal{L}^T\, y &\iff Tx \cup \{x\} = Ty \cup \{y\}, \\ x \,\mathcal{H}^T\, y &\iff x \,\mathcal{R}^T\, y \wedge x \,\mathcal{L}^T\, y. \end{aligned}$$

The *Green index* of T in S is the number of $\mathcal{H}^T$-classes in $S - T$.

Proposition (C, Maltcev)

Let T have finite Green index in S, with $T\phi \subseteq T$. Then $\Gamma(\phi) = \Gamma(\phi|_T)$.

Semigroups with soluble word problem

Theorem (C, Maltcev)

Every computable real number arises as an endomorphism of a finitely generated semigroup with soluble word problem.

Question

What are the growths of endomorphisms of finitely presented semigroups? (Always computable?)

Question

What are the growths of endomorphisms of semigroups presented by finite complete rewriting systems?

Hopficity and co-hopficity

S is hopfian if every surjective homomorphism $\phi : S \to S$ is injective and thus an automorphism.
S is co-hopfian if every injective homomorphism $\phi : S \to S$ is surjective and thus an automorphism.

Theorem (Maltcev, Ruškuc)

Suppose T has finite Rees index in S, and that S and T are finitely generated. If T is hopfian, then S is hopfian as well.

- Key is to prove that $T\phi \neq S$ when $T \neq S$.
- Without finite generation, the result does not hold.
- T does not inherit hopficity from S.

Co-hopficity

Theorem (C, Maltcev)

Suppose T has finite Rees index in S, and that S and T are finitely generated. If T is co-hopfian, then S is co-hopfian as well.

Proof.

Assume T is co-hopfian. Let B generate T. Let $\phi : S \to S$ be an injective endomorphism.
Let $t \in T$. Consider $t\phi$, $t\phi^2$, ...

- If $t\phi^i = t\phi^j$ for $i < j$, then $t\phi^{j-i} = t$ and so $t\phi^{\ell(j-i)} \in T$ for all $\ell \in \mathbb{N}$.
- If $t\phi$, $t\phi^2$, ... are distinct, then $t\phi^\ell \in T$ for $\ell > N$.

In both cases, there exist $k_t, m_t \in \mathbb{N}$ such that $t\phi^{\ell m_t} \in T$ for all $\ell \geqslant k_t$.
Let $k = \max\{k_t : t \in B\}$ and $m = \operatorname{lcm}\{m_t : t \in B\}$. Then $B\phi^{km} \subseteq T$, and so $T\phi^{km} \subseteq T$.

Co-hopficity

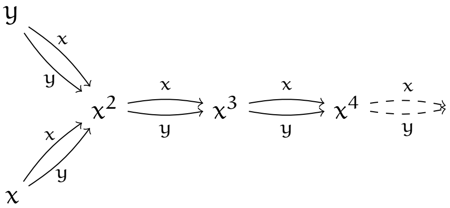
Proof (continued).

Since $\phi : S \to S$ is injective, so is $\phi^{km} : S \to S$.
Hence $\phi^{km}|_T : T \to T$ is injective and so bijective (by co-hopficity of T).
So $\phi^{km}|_{S-T} : S - T \to S - T$ must be injective and so bijective (since $S - T$ is finite).
Thus $\phi^{km} : S \to S$ is bijective, and hence so is ϕ. $\square$

Co-hopficity

Example

Let $S = \langle x, y \mid y^2 = xy = yx = x^2 \rangle$, and let $T = \langle x \rangle = \{x\}^+$. Then $|S - T| = 1$. Then S is co-hopfian but T is not co-hopfian.



Co-hopficity

Example

Let Γ be the graph

$$\begin{array}{ccccccccccccccc}
 & & & & & & & & z_1 & & z_2 & & z_3 & & \\
 & & & & & & & & | & & | & & | & & \\
 & y_{-3} & & y_{-2} & & y_{-1} & & y_0 & & y_1 & & y_2 & & y_3 & \\
 & | & & | & & | & & | & & | & & | & & | & \\
\text{- -} & x_{-3} & — & x_{-2} & — & x_{-1} & — & x_0 & — & x_1 & — & x_2 & — & x_3 & \text{- -}
\end{array}$$

and let $\Delta = \Gamma - \{y_0\}$.

Let $S_\Gamma = \Gamma \cup \{e, n, 0\}$ and define products by

$$v_1 v_2 = \begin{cases} e & \text{if there is an edge between } v_1 \text{ and } v_2 \text{ in } \Gamma, \\ n & \text{if there is no edge between } v_1 \text{ and } v_2 \text{ in } \Gamma, \end{cases}$$

$$v_1 e = e v_1 = v_1 n = n v_1 = en = ne = e^2 = n^2 = 0x = x0 = 0$$

for $v_1, v_2 \in V$ and $x \in S_\Gamma$.

Define S_Δ similarly. Then $|S_\Gamma - S_\Delta| = 1$, S_Γ is not co-hopfian and S_Δ is co-hopfian.

Summary of hopficity and co-hopficity results

	Preserved on passing to			
	Finite Rees index		Finite Green index	
Property	Subsemigroup	Extension	Subsemigroup	Extension
Hopficity	N	N	N	N
Hopficity & f.g.	N	Y	N	N
Co-hopficity	N	N	N	N
Co-hopficity & f.g.	N	Y	N	?

Question

Is ‘co-hopficity and finite generation’ preserved on passing to finite Green index extensions?