# Semigroups with skeletons and Zappa-Szép products

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### The relations $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$

Let $S$ be a semigroup and $E$ be a distinguished set of idempotents. The relation $\widetilde{\mathcal{R}}_E$ is defined by $a \, \widetilde{\mathcal{R}}_E \, b$ if and only if for all $e \in E$,

$$ea = a \Leftrightarrow eb = b.$$

The relation $\widetilde{\mathcal{L}}_E$ is dual.
Note that:

- The relations $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$ are equivalence relations.
- $\mathcal{R} \subseteq \widetilde{\mathcal{R}}_E$ and $\mathcal{L} \subseteq \widetilde{\mathcal{L}}_E$.

The relation $\widetilde{\mathcal{H}}_E$ is the intersection of $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$ and the relation $\widetilde{\mathcal{D}}_E$ is the join of $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$.

# Definitions and basics

A semigroup $S$ satisfies the congruence condition $(C)$ if $\widetilde{\mathcal{R}}_E$ is a left congruence and $\widetilde{\mathcal{L}}_E$ is a right congruence.

We will denote the $\widetilde{\mathcal{R}}_E$-class ($\widetilde{\mathcal{L}}_E$-class, $\widetilde{\mathcal{H}}_E$-class) of any $a \in S$ by $\widetilde{R}_E^a$ ($\widetilde{L}_E^a$, $\widetilde{H}_E^a$).

If $S$ satisfies $(C)$, then $\widetilde{H}_E^e$ is a monoid with identity $e$, for any $e \in E$.

## Weakly $E$-abundant semigroups

A semigroup $S$ with $E \subseteq E(S)$ is said to be *weakly $E$-abundant* if every $\widetilde{\mathcal{R}}_E$- and every $\widetilde{\mathcal{L}}_E$-class of $S$ contains an idempotent of $E$.

## $E$-regular elements

Let $S$ be a semigroup and $E \subseteq E(S)$. We say that an element $c \in S$ is *$E$-regular* if $c$ has an inverse $c^\circ$ such that $cc^\circ, c^\circ c \in E$.

# Analogues of Green's Lemmas

**Lemma** Let $S$ be a semigroup with $(C)$ and suppose $S$ has an $E$-regular element $c$ such that

$$cc^\circ = e\,,\; c^\circ c = f$$

Then the right translations

$$\rho_c : \widetilde{L}_E^e \to \widetilde{L}_E^f \quad \text{and} \quad \rho_{c^\circ} : \widetilde{L}_E^f \to \widetilde{L}_E^e$$

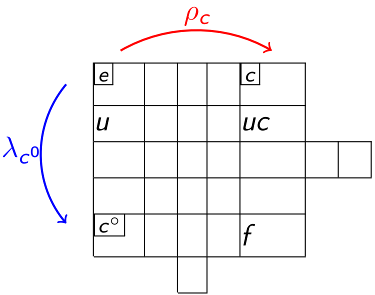
are mutually inverse $\widetilde{\mathcal{R}}_E$-class preserving bijections and the left translations

$$\lambda_{c^\circ} : \widetilde{R}_E^e \to \widetilde{R}_E^f \quad \text{and} \quad \lambda_c : \widetilde{R}_E^f \to \widetilde{R}_E^e$$

are mutually inverse $\widetilde{\mathcal{L}}_E$-class preserving bijections.

# Analogues of Green's Lemmas

The following "egg" box picture helps us to understand the above Lemma




**Corollary** Let $S$ be a semigroup with (C). Let $c$ be an $E$-regular element of $S$ such that

$$cc^\circ = e\,,\; c^\circ c = f\,.$$

Then $\widetilde{H}_E^e \cong \widetilde{H}_E^f$.

# Observations

Let $S$ be a semigroup and $E \subseteq E(S)$. Suppose every $\widetilde{\mathcal{H}}_E$-class contains an $E$-regular element. Then

1. $S$ is weakly $E$-abundant;
2. if $S$ has $(C)$, then $\widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$ (so that $\widetilde{\mathcal{D}}_E = \widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E$);
3. if $a, b \in S$ with $a \, \widetilde{\mathcal{D}}_E \, b$, then $|\widetilde{H}_E^a| = |\widetilde{H}_E^b|$;
4. if $E$ is a band and $\widetilde{\mathcal{H}}_E$ is a congruence, then for $k \in S$ and $k \, \widetilde{\mathcal{H}}_E \, k^2$, $E \cap \widetilde{H}_E^k \neq \emptyset$.

# Semigroups inheriting congruence extension property

A subsemigroup $M$ of a semigroup $S$ has the right congruence extension property if for any right congruence $\rho$ on $M$ we have

$$\rho = \bar{\rho} \cap (M \times M)$$

where $\bar{\rho} = \langle \rho \rangle$ is right congruence on $S$.

**Lemma** Let $S$ be a weakly $E$-abundant semigroup with $(C)$. Suppose that $\widetilde{\mathcal{H}}_E$ is a congruence. Let $e \in E$. Then $M = \widetilde{H}_E^e$ has the right congruence extension property.

# Semigroups inheriting congruence extension property

We say that a congruence $\rho$ on $M$ is closed under conjugation if for $u, v \in M$ with $u \, \rho \, v$ and for any $c \in S$, with $cc^\circ, c^\circ c \in E$ and $cuc^\circ, cvc^\circ \in M$,

$$cuc^\circ \, \rho \, cvc^\circ$$

**Lemma** Let $S$ be a semigroup with $(C)$ such that every $\widetilde{\mathcal{H}}_E$-class contains an $E$-regular element, $E$ is a band and $\widetilde{\mathcal{H}}_E$ is a congruence. Let $e \in E$ and $M = \widetilde{H}_E^e$. Let $\rho$ be a congruence on $M$. Then

$$\rho = \bar{\rho} \cap (M \times M)$$

if and only if $\rho$ is closed under conjugation.

# Restriction semigroups

Left restriction semigroups form a variety of unary semigroups, that is, semigroups equipped with an additional unary operation, denoted by $^+$. The identities that define a left restriction semigroup $S$ are:

$$a^+a = a, a^+b^+ = b^+a^+, (a^+b)^+ = a^+b^+, ab^+ = (ab)^+a.$$

We put

$$E = \{a^+ : a \in S\},$$

then $E$ is a semilattice known as the semilattice of projections of $S$.

Dually right restriction semigroups form a variety of unary semigroups. In this case the unary operation is denoted by $^*$.

A semigroup is restriction, if it is both left and right restriction with same semilattice of projections.

If a restriction semigroup $S$ has an identity element 1, then

$$1^+ = 1^* = 1.$$

Such a restriction semigroup is called a restriction monoid.

We consider special classes of restriction semigroups that consists of single $\widetilde{\mathcal{D}}_E$-classes. Such semigroups are called $\widetilde{\mathcal{D}}_E$-simple semigroups.

# $\widetilde{\mathcal{H}}_E$-transversal subsets

We say that a subset $V$ of $W$, where $W \subseteq S$ and $W$ is a union of $\widetilde{\mathcal{H}}_E$-classes, is an *$\widetilde{\mathcal{H}}_E$-transversal* of $W$ if

$$|V \cap \widetilde{H}_E^a| = 1 \quad \text{for all } a \in W.$$

Example1

Let $S = BR(M, \theta)$, where $M$ is a monoid. Then $(0, 1, 0)$ is the identity of $S$ and

$$\widetilde{L}_E^{(0,1,0)} = \{(a, l, 0) : a \in \mathbb{N}^0, \, l \in M\},$$

$$\widetilde{R}_E^{(0,1,0)} = \{(0, m, a) : a \in \mathbb{N}^0, \, m \in M\}$$

are $\widetilde{\mathcal{L}}_E$- and $\widetilde{\mathcal{R}}_E$-classes of the identity respectively. Let

$$L = \{(a, 1, 0) : a \in \mathbb{N}^0\}.$$

Clearly $L$ is a submonoid $\widetilde{\mathcal{H}}_E$-transversal of $\widetilde{L}_E^{(0,1,0)}$ .

# Inverse skeleton

**Definition**

Let $S$ be a semigroup with $E \subseteq E(S)$. Let $U$ be a subset of $S$ consisting of $E$-regular elements, where $E \subseteq U$. If $U$ intersects every $\widetilde{\mathcal{H}}_E$-class of $S$ ($U$ is an $\widetilde{\mathcal{H}}_E$-transversal of $S$), then $U$ is a *(combinatorial) inverse skeleton* of $S$. If in addition $U$ is a subsemigroup, then $U$ is a *(combinatorial) inverse $S$-skeleton.*

**Example**

Let $S = \mathcal{B}^{\circ}(M, I)$ be a Brandt semigroup, where $M$ is a monoid. Then

$$U = \{(i, 1, j) : i \in I\} \cup \{0\}$$

is a combinatorial inverse $S$-skeleton of $S$.

**Theorem 1** Let $S$ be a $\widetilde{\mathcal{D}}_E$-simple restriction monoid with $\widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$. Suppose there is a submonoid $\widetilde{\mathcal{H}}_E$-transversal $L$ of $\widetilde{L}_E^1$ such that every $c \in L$ is $E$-regular and for all $c \in L$, $e \in E$ we have $cec^\circ, c^\circ ec \in E$. Let

$$R = \{c^\circ : c \in L\}.$$

Then $R$ is a submonoid $\widetilde{\mathcal{H}}_E$-transversal of $\widetilde{R}_E^1$.

Suppose in addition that $RL \subseteq R \cup L$. Then $U = \langle R \cup L \rangle = LR$ and $U$ is a combinatorial inverse $S$-skeleton for $S$.

### Example

Going back to Example 1 let $(a, 1, 0) \in L$. Putting

$$(a, 1, 0)^\circ = (0, 1, a)$$

we have that $(a, 1, 0)^\circ$ is an inverse of $(0, 1, a)$. Set

$$R = \{(a, 1, 0)^\circ : (a, 1, 0) \in L\}$$

We note that $R$ is a submonoid $\widetilde{\mathcal{H}}_E$ transversal of $\widetilde{R}_E^{(0,1,0)}$. Also $RL \subseteq R \cup L$.

Then

$$U = \{(a, 1, b) : a, b \in \mathbb{N}^0\}$$

is a combinatorial inverse $S$-skeleton of $S$.

Example

Let $S = BR(M, \mathbb{Z}, \theta)$ be extended Bruck-Reilly extension of monoid $M$. The semigroup operation on $S$ is defined by the rule:

$$(k, s, l)(m, t, n) = \begin{cases} (k - l + m, (s)\theta^{m-l}t), n), & \text{if } l < m; \\ (k, st, n), & \text{if } l = m; \\ (k, s(t)\theta^{l-m}, n - m + l), & \text{if } l > m. \end{cases}$$

for $k, l, m, n \in \mathbb{Z}$ and $s, t \in M$. Then $S$ has an inverse skeleton

### Example

Let $S = [Y; S_\alpha, \chi_{\alpha,\beta}]$ be a strong semilattice $Y$ of monoids $S_\alpha$, where

$$\chi_{\alpha,\beta} : S_\alpha \to S_\beta$$

is a monoid homomorphism such that

1. $\chi_{\alpha,\alpha} = 1_{S_\alpha}$,
2. $\chi_{\alpha,\beta}\chi_{\beta,\gamma} = \chi_{\alpha,\gamma}$ if $\alpha \geq \beta \geq \gamma$

On $S = \cup_{\alpha \in Y} S_\alpha$, multiplication is defined by

$$ab = (a\chi_{\alpha,\alpha\beta})(b\chi_{\beta,\alpha\beta}) \quad a \in S_\alpha,\ b \in S_\beta.$$

Let $e_\alpha$ be the identity of $S_\alpha$. Then $E = \{e_\alpha : \alpha \in Y\}$ is a semilattice, $S$ is a restriction semigroup with respect to $E$ and the $\widetilde{\mathcal{H}}_E$-classes are the $S_\alpha$'s. Then $E$ is an inverse $S$-skeleton.

# Special $\widetilde{\mathcal{D}}_E$-simple restriction monoids

Definition

Let $S$ be a $\widetilde{\mathcal{D}}_E$-simple restriction monoid. We say that $S$ is special if $\widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$ and there is a submonoid $\widetilde{\mathcal{H}}_E$-transversal $L$ of $\widetilde{L}^1_E$ such that every $c \in L$ is $E$-regular and for all $c \in L$, $e \in E$ we have $cec^\circ, c^\circ ec \in E$.

If $S$ is a special $\widetilde{\mathcal{D}}_E$-simple restriction monoid, then by Theorem 1 $R = \{c^\circ : c \in L\}$ is a submonoid $\widetilde{\mathcal{H}}_E$-transversal of $\widetilde{R}^1_E$.

Let $S$ and $T$ be semigroups and suppose that we have maps

$$T \times S \to S, \quad (t, s) \mapsto t \cdot s$$
$$T \times S \to T, \quad (t, s) \mapsto t^s$$

such that for all $s, s' \in S, t, t' \in T$, the following hold:

ZS1 $tt' \cdot s = t \cdot (t' \cdot s)$;

ZS2 $t \cdot (ss') = (t \cdot s)(t^s \cdot s')$;

ZS3 $(t^s)^{s'} = t^{ss'}$;

ZS4 $(tt')^s = t^{t' \cdot s} t'^s$.

Define a binary operation on $S \times T$ by

$$(s, t)(s', t') = (s(t \cdot s'), t^{s'} t').$$

# Zappa-Szép products

Then $S \times T$ is a semigroup, known as the Zappa-Szép product of $S$ and $T$ and denoted by $S \bowtie T$.

If $S$ and $T$ are monoids then we insist that the following four axioms also hold:

ZS5 $t \cdot 1_S = 1_S$;

ZS6 $t^{1_S} = t$;

ZS7 $1_T \cdot s = s$;

ZS8 $1_T^s = 1_T$.

Then $S \bowtie T$ is monoid with identity $(1_S, 1_T)$.

# The Bruck-Reilly extension of a monoid

*Kunze* discovered that the Bruck-Reilly extension of a monoid $BR(S,\theta)$ is the Zappa-Szép product of $\mathbb{N}^0$ under addition and the semidirect product $\mathbb{N}^0 \ltimes S$, where multiplication in $\mathbb{N}^0 \ltimes S$ is defined by the following rule:

$$(k,s)\cdot(l,t) = (k+l,(s\theta^l)t).$$

Define for $m \in \mathbb{N}^0$ and $(l,s) \in \mathbb{N}^0 \ltimes S$

$$m\cdot(l,s) = (g-m, s\theta^{g-l}) \text{ and } m^{(l,s)} = g-l$$

where $g$ is greater of $m$ and $l$. Then $(\mathbb{N}^0 \ltimes S) \times \mathbb{N}^0$ is Zappa-Szép product with composition rule

$$[(k,s),m]\circ[(l,t),n] = [(k-m+g, s\theta^{g-m}t\theta^{g-l}), n-l+g],$$

where again $g$ is greater of $m$ and $l$.

# Special $\widetilde{\mathcal{D}}_E$-simple restriction monoids and Zappa-Szép products

**Theorem 2** Let $S$ be a special $\widetilde{\mathcal{D}}_E$-simple restriction monoid. Then $M = L \bowtie \widetilde{R}_E^1$ is a Zappa-Szép product of $L$ and $\widetilde{R}_E^1$ under the actions defined by

$$r \cdot l = d \text{ where } d \in L \text{ and } d^+ = (rl)^+$$

and

$$r^l = d^\circ rl \text{ where } d \in L \text{ and } d^+ = (rl)^+$$

for $l \in L$ and $r \in \widetilde{R}_E^1$. Further $S \cong M$.

# Special $\widetilde{\mathcal{D}}_E$-simple restriction monoids and Zappa-Szép products

We explain these actions with the help of an egg box picture.

| | | | |
|---|---|---|---|
| $\boxed{1}$ | | $r$ | $r^l = d^\circ rl$ |
| $\boxed{l}$ | | | |
| $\boxed{r \cdot l = d}$ | | | $rl$ |
| | | | |

# Special $\widetilde{\mathcal{D}}_E$-simple restriction monoids and Zappa-Szép products

**Theorem 3** Let $S$ be a special $\widetilde{\mathcal{D}}_E$-simple restriction monoid. Then $Z = \widetilde{H}_E^1 \bowtie R$ is a Zappa-Szép product isomorphic to $\widetilde{R}_E^1$ under the action of $R$ on $\widetilde{H}_E^1$ defined by

$$r \cdot h = rht^\circ \text{ where } t^* = (rh)^* \text{ and } t \in R.$$

and action of $\widetilde{H}_E^1$ on $R$ by

$$r^h = t \text{ where } t^* = (rh)^* \text{ and } t \in R.$$

Now we see that if $\widetilde{\mathcal{H}}_E$ is a congruence, then for $r \in R$ and $h \in \widetilde{H}_E^1$

$$rh \, \widetilde{\mathcal{H}}_E \, r1 = r$$

and thus $r^h = r$, so that $Z$ becomes a semidirect product.

# Deduction

Kunze showed that if $S$ is a monoid and $\mathbb{N}$ is the set of natural numbers under addition, then a semidirect product $\mathbb{N}^0 \ltimes S$ can be formed under the multiplication,

$$(k, s)(l, t) = (k + l, (s\theta^l)t).$$

Now we see that

$$L_1 = \{(l, s, 0) : l \in \mathbb{N}^0, s \in S\},$$

so that if we put

$$L = \{(l, e, 0) : l \in \mathbb{N}^0\} \cong \mathbb{N}^0,$$

then $L$ is submonoid $\widetilde{\mathcal{H}}_E$-transversal of $L_1$. Further,

$$\widetilde{H}_1 = \{(0, s, 0) : s \in S\}.$$

# Deduction

For $(l, e, 0) \in L$ and $(0, s, 0) \in \widetilde{H}_1$,

$$\begin{aligned}(0, s, 0)^{(l,e,0)} &= (l, e, 0)^{-1}(0, s, 0)(l, e, 0)\\ &= (0, s\theta^l, 0) \in \widetilde{H}_1.\end{aligned}$$

Thus $L \ltimes \widetilde{H}_1$ is semidirect product under multiplication defined by

$$\big((k, e, 0), (0, s, 0)\big)\big((l, e, 0), (0, t, 0)\big) = \big((k + l, e, 0), (0, s\theta^l t, 0)\big).$$

# Applications to bisimple inverse monoids

We specialise Theorem 2 and Theorem 3 to obtain corresponding results for bisimple inverse monoids.

### Example

The bicyclic semigroup $B$ is the Zappa-Szép product of $L = L_1$ and $R = R_1$, where

$$L = \{(m, 0) : m \in \mathbb{N}^0\} \cong \mathbb{N}^0$$

$$R = \{(0, n) : n \in \mathbb{N}^0\} \cong \mathbb{N}^0$$

under the actions of $R$ on $L$ and $L$ on $R$ defined respectively as:

$$(0, m) \cdot (n, 0) = (\max(m, n) - m, 0)$$

and

$$(0, m)^{(n,0)} = (0, \max(m, n) - n).$$


Thank You