K-Theory of Inverse Semigroups

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Background

Dualities in mathematics:

- Order structures and discrete spaces (Stone duality)
- Locally compact Hausdorff spaces and commutative C^*-algebras (Gelfand representation theorem)

Also, (von Neumann) regular rings similar to regular semigroups

Background

Paterson (1980's) and Renault (1980) generalised this to deep connections between 3 different "discrete" mathematical structures:

- Inverse semigroups (generalised order structures)
- Topological groupoids (generalised discrete spaces)
- C^*-algebras

Background

Some successful applications:

- Topological K-theory and operator / algebraic K-theory (Serre-Swan theorem)
- Module theory for rings (Dedekind + others) and act theory for monoids
- Morita equivalence of semigroups (Knauer, Talwar) vs. Morita equivalence for rings (Morita)
- Morita equivalence for inverse semigroups (Afara, Funk, Laan, Lawson, Steinberg) vs. Morita equivalence for C^*-algebras (Rieffel + others)

Examples

- Polycyclic / Cuntz monoid / Cuntz groupoid / Cuntz algebra
- Graph inverse semigroups / Cuntz-Krieger semigroups / Cuntz-Krieger groupoid / Cuntz-Krieger algebra
- Boolean inverse monoids / Boolean groupoids
- Tiling semigroups / tiling groupoids / tiling C^*-algebras

Grothendieck group

Theorem

Let S be a commutative semigroup. Then there is a unique (up to isomorphism) commutative group $G = \mathcal{G}(S)$, called the *Grothendieck group*, and a homomorphism $\phi : S \to G$, such that for any commutative group H and homomorphism $\psi : S \to H$, there is a unique homomorphism $\theta : G \to H$ with $\psi = \theta \circ \phi$.

Algebraic K-theory

- R - ring
- $\mathbf{Proj}_R$ - finitely generated projective modules of R.
- $(\mathbf{Proj}_R, \oplus)$ is a commutative monoid.
- Define
$$K_0(R) = \mathcal{G}(\mathbf{Proj}_R).$$
- If X is a compact Hausdorff space and $C(X)$ is the ring of $\mathbb{F}$-valued continuous functions on X then
$$K^0_{\mathbb{F}}(X) \cong K_0(C(X)).$$

Idempotent matrices

- Let $M(R)$ denote the set of $\mathbb{N}$ by $\mathbb{N}$ matrices over R with finitely many non-zero entries.
- Idempotent matrices correspond to projective modules
- Say idempotent matrices $E, F \in M(R)$ are similar and write $E \sim F$ if $E = XY$ and $F = YX$ where $X, Y \in M(R)$.

Proposition

Idempotent matrices E and F define the same projective module if and only if $E \sim F$.

Idempotent matrices

- Denote the set of idempotent matrices by $\mathrm{Idem}(R)$ and define a binary operation on $\mathrm{Idem}(R)/\sim$ by

$$[E] + [F] = [E' + F'],$$

where if a row in E' has non-zero entries then that row in F' has entries only zeros, similarly for columns of E', and for rows and columns of F', and such that $E' \sim E$ and $F' \sim F$.

Theorem

This is a well-defined operation and the monoids $\mathrm{Idem}(R)/\sim$ and $\mathbf{Proj}_R$ are isomorphic.

- This gives us an alternative way of viewing $K_0(R)$:

$$K_0(R) = \mathcal{G}(\mathrm{Idem}(R)/\sim).$$

K-theory of inverse semigroups

- Idea: want to define $K_0(S)$ for S an inverse semigroup.
- Need to restrict the class of inverse semigroups - will not be a problem.
- Give definition in terms of projective modules and definition in terms of idempotent matrices.
- Want $K_0(S) \cong K_0(C(S))$, where $C(S)$ is some C^*-algebra associated to S.

Some inverse semigroup theory

- An *inverse semigroup* is a semigroup S such that for every element $s \in S$ there exists a unique element $s^{-1} \in S$ with $ss^{-1}s = s$ and $s^{-1}ss^{-1} = s^{-1}$ (without uniqueness, we have a *regular semigroup*).
- A regular semigroup is inverse if and only if its idempotents commute.
- Natural partial order (NPO): $s \leq t$ iff $s = ts^{-1}s$.
- Remark: the set of idempotents form a meet semilattice under the operation $e \wedge f = ef$.

Orthogonally complete inverse semigroups

- Firstly, we will assume our inverse semigroup S has a *zero* $(0s = s0 = 0)$.
- Next, we want our inverse semigroup to be sufficiently ring like, namely we require *orthogonal completeness* - this will not be a problem as every inverse semigroup with 0 has an *orthogonal completion* and the examples we are interested in are orthogonally complete.
- Elements $s, t \in S$ are *orthogonal*, written $s \perp t$, if $st^{-1} = s^{-1}t = 0$.
- S is *orthogonally complete* if
  1. $s \perp t$ implies there exists $s \vee t$
  2. $s \perp t$ implies $u(s \vee t) = us \vee ut$ and $(s \vee t)u = su \vee tu$.

Rook matrices

- Throughout what follows S will be an orthogonally complete inverse semigroup.
- A matrix A with entries in S is said to be a *rook matrix* if it satisfies the following conditions:
  1. (RM1): If a and b lie in the same row of A then $a^{-1}b = 0$.
  2. (RM2): If a and b lie in the same column of A then $ab^{-1} = 0$.
- $R(S)$ = all finite-dimensional rook matrices
- $M_n(S)$ = all $n \times n$ matrices
- $M_\omega(S) = \mathbb{N} \times \mathbb{N}$ rook matrices with finitely many non-zero entries.

Facts about rook matrices

- $R(S)$ is an inverse semigroupoid.
- $M_n(S)$ and $M_\omega(S)$ are orthogonally complete inverse semigroups.
- Let
$$A(S) = E(M_\omega(S))/\mathcal{D}.$$
- Define $[E] + [F] = [E' \vee F']$.
- We will define
$$K(S) = \mathcal{G}(A(S)).$$
- $S \mapsto M_\omega(S)$ and $S \mapsto K(S)$ have functorial properties.

Pointed étale sets

A *pointed étale set* is a set X together with a right action of S on X, a map $p : X \to E(S)$ and a distinguished element 0 satisfying the following:

- $x \cdot p(x) = x$.
- $p(x \cdot s) = s^{-1}p(x)s$.
- $p(0_X) = 0$ and if $p(x) = 0$ then $x = 0_X$.
- $0_X \cdot s = 0_X$ for all $s \in S$.
- $x \cdot 0 = 0_X$ for all $x \in X$.

Define a partial order on X: $x \leq y$ iff $x = y \cdot p(x)$.
Define $x \perp y$ if $p(x)p(y) = 0$ and say that x and y are *orthogonal*.
We will say elements $x, y \in X$ are *strongly orthogonal* if $x \perp y$, $\exists x \vee y$ and $p(x) \vee p(y) = p(x \vee y)$.

Premodules and modules

A *premodule* is a pointed étale set such that

- If $x, y \in X$ are strongly orthogonal then for all $s \in S$ we have $x \cdot s$ and $y \cdot s$ are strongly orthogonal and $(x \vee y) \cdot s = (x \cdot s) \vee (y \cdot s)$.
- If $s, t \in S$ are orthogonal then $x \cdot s$ and $x \cdot t$ are strongly orthogonal for all $x \in X$.

A *module* is a pointed étale set such that

- If $x \perp y$ then $\exists x \vee y$ and $p(x \vee y) = p(x) \vee p(y)$.
- If $x \perp y$ then $(x \vee y) \cdot s = x \cdot s \vee y \cdot s$.

Examples

- 0 is a module with $0 \cdot s = 0$ for all $s \in S$ (initial object in category).
- eS is a premodule with $es \cdot t = est$ and $p(es) = s^{-1}es$.
- S itself is a premodule with $s \cdot t = st$ and $p(s) = s^{-1}s$.
- In fact, every right ideal is a premodule.

Categories

- We will define premodule morphisms and module morphisms $f : (X, p) \to (Y, q)$ to be structure preserving maps between, respectively, premodules and modules.
- Note that we require $q(f(x)) = p(x)$.
- We denote the category of premodules of S by $\mathbf{Premod}_S$ and modules by $\mathbf{Mod}_S$.
- Monics in $\mathbf{Premod}_S$ and $\mathbf{Mod}_S$ are injective and epics in $\mathbf{Mod}_S$ are surjective.
- $\mathbf{Mod}_S$ is cocomplete.

Proposition

There is a functor $\mathbf{Premod}_S \to \mathbf{Mod}_S$, $X \mapsto X^\sharp$, which is left adjoint to the forgetful functor.

Coproducts

Can define coproduct in $\mathbf{Mod}_S$ for (X, p), (Y, q) by

$$X \bigoplus Y = \{(x, y) \in X \times Y | p(x)q(y) = 0\}$$

with

$$(p \oplus q)(x, y) = p(x) \vee q(y)$$

and

$$(x, y) \cdot s = (x \cdot s, y \cdot s).$$

Projective modules

- A projective module P is one such that for all morphisms $f : P \to Y$ and epics $g : X \to Y$ there is a map $h : P \to X$ with $gh = f$.
- If P_1, P_2 projective then $P_1 \bigoplus P_2$ is projective.
- $(eS)^\sharp$ is projective.
- Denote by $\mathbf{Proj}_S$ the category of modules X with

$$X \cong \bigoplus_{i=1}^{m} (e_i S)^\sharp.$$

Theorem

Let $\mathbf{e} = (e_1, \ldots, e_m)$, $\mathbf{f} = (f_1, \ldots, f_n)$ and $\Delta(\mathbf{e}), \Delta(\mathbf{f})$ be the associated diagonal matrices in $M_\omega(S)$. Then

$$\bigoplus_{i=1}^{m} (e_i S)^\sharp \cong \bigoplus_{i=1}^{n} (f_i S)^\sharp$$

if, and only if,

$$\Delta(\mathbf{e}) \, \mathcal{D} \, \Delta(\mathbf{f}).$$

Corollary

$$K(S) = \mathcal{G}(\mathbf{Proj}_S).$$

- Can define *states* and *traces* on S
- If S commutative, then $K(S) \cong K(E(S))$.
- If S commutative or nice then can form tensor products of matrices and modules - sometimes gives a ring structure on $K(S)$.

Examples

- Symmetric inverse monoids:

$$K(I_n) = \mathbb{Z}.$$

- (Unital) Boolean algebras:

$$K(A) = K^0(S(A)).$$

- Cuntz-Krieger semigroups:

$$K(CK_{\mathcal{G}}) = K^0(\mathcal{O}_{\mathcal{G}}).$$

Thank you for listening