Sierpiński Rank and Universal Sequences

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Definintion of Sierpiński Rank

Definition

The *Sierpiński rank* of a semigroup S is the least n (possibly ∞) such that any countable subset of S is contained in an n-generated subsemigroup of S.

Countable $\implies$ (rank = Sierpiński rank)

Another way of looking at Sierpiński rank is the smallest n such that any countable subset of S is contained in the image of some homomorphism from F_n the free semigroup on n generators.

Sierpiński rank of the Transformation Monoid 1

Theorem (Sierpiński)

The Sierpiński rank of the transformation monoid on a countable set is 2.

Let $(a_n)_{n \in \mathbb{N}_0}$ a sequence of functions on S the set of eventually constant sequences over $\mathbb{N}_0$ be given.

- Let f be the injection on S which appends 0 to the beginning of the sequence
- Let g be the injection on S which increments the first term by 1
- Let h be the function which sends $(x_1, x_2, x_3, \dots)$ to $(x_2, x_3, \dots)a_{x_1}$.

The Sierpiński rank of the Transformation Monoid 2

For all n

$$a_n = fg^n h$$

We wanted 2 generators and we have 3. But when $n = 0$ we have

$$a_0 = fh$$

where we do not use g at all. So assume (WLOG) that $a_0 = g$

$$a_n = f(fh)^n h$$

So our countable set (the terms of the sequence $(a_n)_n$) is now contained in

$$\langle f, h \rangle$$

a two generated subsemigroup, as desired.

Universal Sequences

Definition

A sequence $(x_i)_i$ over some free semigroup F_n is a *universal sequence* for a semigroup S if for any sequence $(y_i)_i$ over S there is a homomorphism from F_n to S which for each i maps x_i to y_i.

A homomorphism from F_n is of course defined by the image of its generators. So we could equivalently think of a universal sequence as a sequence of products over some free variables such that we can get any sequence over S by substituting in different tuples over S.

Examples of Universal Sequences

Some universal sequences for $T_{\mathbb{N}}$: (shamelessly plagarised from James Mitchell's slides)

- $\left(a^2b^3(abab^3)^{i+1}ab^2ab^3\right)_i$ (Sierpiński)
- $\left(aba^{i+1}b^2\right)_i$ (Banach)
- $\left(abab^{i+3}ab^2\right)_i$ (Hall)
- $\left(aba^{i+2}b^{i+2}\right)_i$ (Mal'cev)
- $\left(a^2b^{i+2}ab\right)_i$ (McNulty)
- $\left(a(ab)^ib\right)_i$ (J.H, J.M, Y.P)

We have shown $\left(aaaa(aab)^ia(aab)^ibb\right)_i$ is a universal sequence for the monoid of partial bijections.

Galvin showed that $\left(a^{-1}(a^iba^{-i})b^{-1}(a^ib^{-1}a^{-i})ba\right)_i$ is a (group) universal sequence for $\mathrm{Sym}(\mathbb{N})$.

Universal Sequence Rank

Definition

The *universal sequence rank* of a semigroup S is the least n (possibly infinite) such that S has some universal sequence over F_n.

Universal sequence rank $\geq$ Sierpiński rank

The following semigroups have semigroup universal sequence rank 2

- The transformation monoid on any infinite set (Sierpiński)
- The symmetric group on any infinite set (Galvin)
- The monoid of partial bijections on any infinite set (J.H, J.M, Y.P)
- The partition monoid on any infinite set (J.H, J.M, Y.P)
- The endomorphisms of the random graph (J.H, J.M, Y.P)
- The order automorphisms of $\mathbb{Q}$ (or $\mathbb{R}$) (J.H, J.M, Y.P)

Universal Sequence Ranks that are not 2

The injections on $\aleph_n$ have infinite universal sequence rank and Sierpiński rank $n + 4$.

The surjections on $\aleph_n$ have infinite universal sequence rank and Sierpiński rank $\frac{n^2}{2} + \frac{9n}{2} + 7$.

The order endomorphisms of $[0, 1]$ (or $\mathbb{Q} \cup [0, 1]$) have universal sequence rank 3 and Sierpiński rank the same.

Properties

- Universal sequences for groups do not satisfy the pumping lemma for context-free languages.
- Universal sequences for inverse semigroups do not satisfy the pumping lemma for regular languages.
- The property of having a particular universal sequence is closed under arbitary direct product.
- Any semigroup with finite universal sequence rank is totally distorted and therefore has the Bergman property.

Questions

- Does there exist a semigroup with finite but non-equal Sierpiński rank and universal sequence rank?
- What is the universal sequence rank of the automorphism group of the random graph?
- What is the universal sequence rank of the automorphism group of the random partial order?
- What is the set of universal sequences for $T_{\mathbb{N}}$?