Cayley Automaton Semigroups

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Definitions

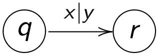
Definition

An *automaton* is a triple $\mathcal{A} = (Q, B, \delta)$ where:

- Q is a finite set of *states*
- B is a finite *alphabet*
- $\delta : Q \times B \to Q \times B$ is the *transition function.*

Definitions cont.

Automata have outputs:



If we are in state q and read symbol x, we move to state r and output y. That is, $\delta(q, x) = (r, y)$.

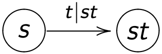
If we're in state q_0 and read a sequence $\alpha_1\alpha_2\ldots\alpha_n$ we output $\beta_1\beta_2\ldots\beta_n$ where $\delta(q_{i-1}, \alpha_i) = (q_i, \beta_i)$.

Starting in state q and reading α gives an endomorphism of the $|B|$-ary rooted tree. Extending this to several states gives a homomorphism $\phi : Q^+ \to End(B^*)$.

We say that $\Sigma(\mathcal{A}) \cong im(\phi)$ is the *automaton semigroup.*

Cayley Automaton Semigroups

$\mathcal{C}(S)$ is the automaton arising from the right Cayley graph of S (where we take all of S as the generating set). A typical edge looks like



More formally:

$$\mathcal{C}(S) = (\overline{S}, S, \delta), \delta(\overline{s}, t) = (\overline{st}, st)$$

where we denote states by $\overline{s}$ to avoid confusion.

$\Sigma(\mathcal{C}(S))$ is the *Cayley Automaton Semigroup*.

How does $\overline{q}$ act on S^*?

Let $x \in S, \alpha \in S^*, \overline{q_i} \in \overline{S}$. Then

$$\overline{q} \cdot (x\alpha) = (qx)(\overline{qx} \cdot \alpha), (\overline{q_1} \cdot \overline{q_2}) \cdot \alpha = \overline{q_1} \cdot (\overline{q_2} \cdot \alpha).$$

For $\alpha = \alpha_1\alpha_2 \dots \alpha_n$ we have

$$\begin{aligned}
\overline{q} \cdot \alpha &= (q\alpha_1)(\overline{q\alpha_1} \cdot \alpha_2 \dots \alpha_n) \\
&= (q\alpha_1)(q\alpha_1\alpha_2)(\overline{q\alpha_1\alpha_2} \cdot \alpha_3 \dots \alpha_2) \\
&\vdots \\
&= (q\alpha_1)(q\alpha_1\alpha_2) \dots (q\alpha_1 \dots \alpha_n)
\end{aligned}$$

So we can think of $\overline{q}$ as a function
$\overline{q} : \alpha_1\alpha_2 \dots \alpha_n \mapsto (q\alpha_1)(q\alpha_1\alpha_2) \dots (q\alpha_1 \dots \alpha_n)$.

Some properties

- (Mintz 2009) Let S be finite. The following are equivalent:
  - S is aperiodic
  - $\Sigma(\mathcal{C}(S))$ is finite
  - $\Sigma(\mathcal{C}(S))$ is aperiodic
- (Silva and Steinberg 2005) Let G be a non-trivial finite group. Then $\Sigma(\mathcal{C}(G)) \cong F_{|G|}$
- (Mintz 2009) Let $T \leq S$. The $\Sigma(\mathcal{C}(T))$ divides $\Sigma(\mathcal{C}(S))$. If T is a non-trivial group then $\Sigma(\mathcal{C}(T)) \leq \Sigma(\mathcal{C}(S))$.

Zeros

Let $z \in S$ be a left-zero. The $\overline{z}$ is a left-zero in $\Sigma(\mathcal{C}(S))$.

$\overline{z} \cdot \alpha = (z\alpha_1)(z\alpha_1\alpha_2)\dots(z\alpha_1\dots\alpha_n) = (z)^n$.Let $a \in S$. Then $\overline{a} \cdot \alpha = \beta_1\beta_2\dots\beta_n$. So $\overline{z} \cdot \overline{a} \cdot \alpha = \overline{z} \cdot \beta_1\beta_2\dots\beta_n = (z)^n$.

Consequently, $\Sigma(\mathcal{C}(L_n)) \cong L_n$ after noting $\overline{y} \cdot \alpha = (y)^n \neq (z)^n = \overline{z} \cdot \alpha$.

Let $0 \in S$ be the zero element. Then $\overline{0}$ is the zero element in $\Sigma(\mathcal{C}(S))$.

Let $z \in S$ be a right zero. Then $\overline{z}$ is a right-zero in $\Sigma(\mathcal{C}(S))$.

Consider R_n. Then $\overline{x} \cdot \alpha = (x\alpha_1)(x\alpha_1\alpha_2)\dots(x\alpha_1\dots\alpha_n) = \alpha_1\alpha_2\dots\alpha_n$ and $\overline{y} \cdot \alpha = \alpha_1\alpha_2\dots\alpha_n$. So $\overline{x} = \overline{y}$ but $x \neq y$.

When does $\overline{x} = \overline{y}$?

Lemma

Let $x \neq y \in S$. Then $\overline{x} = \overline{y} \in \Sigma(\mathcal{C}(S))$ if and only if $xa = ya$ for all $a \in S$.

Proof.

($\Rightarrow$) Let $a\alpha \in S^*$. Then $\overline{x} \cdot a\alpha = (xa)(\overline{xa} \cdot \alpha)$ and $\overline{y} \cdot a\alpha = (ya)(\overline{ya} \cdot \alpha)$. The first symbols of the outputs must be equal and so $xa = ya$ for all $a \in S$.

($\Leftarrow$) Let $xa = ya$. Then $\overline{x} \cdot a\alpha = (xa)(\overline{xa} \cdot \alpha) = (ya)(\overline{ya} \cdot \alpha) = \overline{y} \cdot a\alpha$ and so $\overline{x} = \overline{y}$. □

Nilpotent Semigroups

A semigroup S is *nilpotent of class* n if there exists n such that $S^n = \{0\}$ and $S^{n-1} \neq \{0\}$. Note that such a semigroup must necessarily contain a zero element. By definition a semigroup is nilpotent of class 1 if and only if it is trivial.

Lemma (Cain 2009)

Let S be finite and nilpotent of class n. Then $\Sigma(C(S))$ is finite and nilpotent of class $n-1$.

Proof.

We have $\overline{w_1} \cdot \overline{w_2} \cdot \ldots \cdot \overline{w_{n-1}} \cdot \alpha = (w_1 w_2 \ldots w_{n-1} \alpha_1) \ldots = 0^\omega$ since S is nilpotent of class n. Hence $\Sigma(C(S))$ is nilpotent of class at most $n-1$.

Now let $w_1, \ldots, w_{n-1}$ be such that $w_1 w_2 \ldots w_{n-1} \neq 0$. Then $\overline{w_1} \cdot \ldots \cdot \overline{w_{n-2}} \cdot w_{n-1} = (w_1 w_2 \ldots w_{n-2} w_{n-1}) \neq 0^\omega$. Hence $\overline{w_1} \cdot \ldots \cdot \overline{w_{n-2}} \neq \overline{0}$. So $\Sigma(C(S))$ is nilpotent of class $n-1$. □

Other known classes of Semigroups

Lemma (M 2012)

Let S be cancellative (and not necessarily finite). Then $\Sigma(C(S))$ is free of rank equal to the order of S.

Lemma (M 2011)

Let S be a finite monogenic semigroup with a non-trivial subgroup. Then $\Sigma(C(S))$ is a small extension of a free semigroup of rank equal to the order of the subgroup.

Lemma (Maltcev 2008)

Let S be finite. Then $\Sigma(C(S))$ is free if and only if the minimal ideal K of S consists of a single $\mathcal{R}$-class in which every $\mathcal{H}$-class is non-trivial and there exists k such that $st = skt$ for all $s, t \in S$.

Self-Automaton Semigroups

S is *self-automaton* if $S \cong \Sigma(C(S))$. We are particularly interested in the map $s \mapsto \overline{s}$. Known examples:

- A monoid is self automaton if and only if it is a band
- Left-zero semigroups
- Semilattices
- Zero-unions of left-zero semigroups
- $L_n \cup B$ where L_n acts trivially on the band B

Theorem

Let B be a band. Then the map $b \mapsto \overline{b}$ is a homomorphism.

We can classify which bands are self-automaton.

Theorem (M 2012)

Let B be a band. Then $B \cong \Sigma(C(B))$ under the map $b \mapsto \overline{b}$ if and only if the left-regular representation of B is faithful.

So are all self-automaton semigroups bands? NO!

$S = \langle e, f, a, 0 | e^2 = ef = e, f^2 = fe = f, ae = af = a, ea = fa = a^2 = 0 \rangle$ is self-automaton.

It remains an open question to classify the self-automaton semigroups.

Thanks for listening!