A four element semigroup that is inherently nonfinitely based?

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Finite basis properties

A (finite) algebra A is finitely based if all the identities satisfied by A are consequences of some finite set of such identities.

Otherwise A is nonfinitely based.

It is inherently nonfinitely based if, moreover, any locally finite variety that contains A is also nonfinitely based. In that case, any finite algebra B such that $\mathcal{V}(B)$ contains A is also nonfinitely based.

(A variety is locally finite if each of its finitely generated members is finite. Any finite algebra generates a locally finite variety.)

Finite basis properties for ‘plain’ semigroups

- Every semigroup of fewer than six elements is finitely based
- The six-element Brandt monoid B_2^1 is inherently nonfinitely based. In fact, there is an algorithm to decide whether a finite semigroup is inherently nonfinitely based (Sapir).

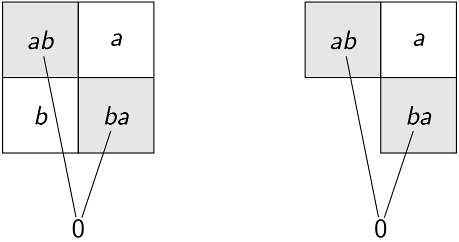
Inverse semigroups, as unary semigroups

- It is known that every inverse semigroup of fewer than six elements is finitely based.
- There are no inherently nonfinitely based inverse semigroups (Sapir).
- But it is conjectured that any finite inverse semigroup such that $\mathcal{V}(\mathcal{S})$ contains B_2^1 is nonfinitely based.

The semigroups B_2 and B_0

- B_2 is the five-element combinatorial, completely 0-simple inverse semigroup. As a semigroup, it may be presented as $\langle a, b \mid aba = a, bab = b, a^2 = b^2 = 0\rangle$.
- B_0 is the subsemigroup $\{a, ab, ba, 0\}$ of B_2.

In terms of Green's relations (shaded boxes depict $\mathcal{H}$-classes containing idempotents):



Bases of identities as 'plain' semigroups

[Trahtman] As a 'plain' semigroup, a basis for the identities of B_2 is:

$$x^3 = x^2, \quad xyx = xyxyx, \quad x^2y^2 = y^2x^2.$$

[Edmunds] As a 'plain' semigroup, a basis of identities for B_0 is:

$$x^3 = x^2, \quad xyx = yxy = (xy)^2 = x^2y^2$$

The varieties they generate have been studied intensively as part of the recent interest in Rees-Sushkevich varieties.

B_2, regarded as an inverse semigroup

As an inverse semigroup, a basis of identities for B_2 is:

$$yxy^{-1} = (yxy^{-1})^2$$

It generates the variety of *combinatorial*, *strict* inverse semigroups.

An inverse semigroup is strict if it satisfies $\mathcal{D}$-majorization: no idempotent is above distinct, $\mathcal{D}$-related idempotents.

An inverse semigroup is completely semisimple if it contains no distinct, comparable $\mathcal{D}$-related idempotents.

Ancient history:

- An inverse semigroup is strict if and only if it is a subdirect product of Brandt semigroups and groups.
- An inverse semigroup is completely semisimple if and only if each principal factor is a Brandt semigroup or group, and if and only if it contains no bicyclic subsemigroup.

(A Brandt semigroup is a completely 0-simple inverse semigroup.)

Restriction semigroups

Intuitively – for the purposes of this talk –

- forget the inverse operation x^{-1} in inverse semigroups
- retain only the induced operations $x^+ = xx^{-1}$ and $x^* = x^{-1}x$.
- the restriction semigroups form the variety of biunary semigroups $(S, \cdot, ^+, ^*)$ generated by the (reducts of) inverse semigroups in this way.

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Identities defining restriction semigroups

The restriction semigroups are defined by the identities

$$x^+x = x; \quad (x^+y)^+ = x^+y^+; \quad x^+y^+ = y^+x^+; \quad xy^+ = (xy)^+x,$$

and their 'duals' (obtained by replacing $^+$ by * and reversing the order of each expression) along with $(x^+)^* = x^+$ and $(x^*)^+ = x^*$.

The set $P_S = \{x^+ : x \in S\}$ is the semilattice of projections of S.

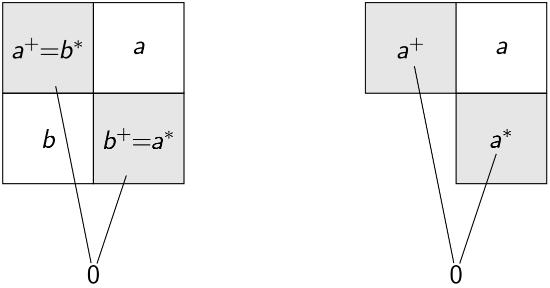
Evolution in the language of the 'York' school:

$$\begin{array}{c}\text{Type-A adequate} = \text{ample} \\ \downarrow \\ \text{weakly ample} \\ \downarrow \\ \text{weakly } E\text{-ample} = \text{restriction}\end{array}$$

In fact, all the specific examples in this talk will actually be ample semigroups. They will be full subsemigroups of Munn semigroups on semilattices.

B_2 and B_0 as restriction semigroups

B_0 has the natural structure of a restriction semigroup, inherited from the inverse semigroup B_2.

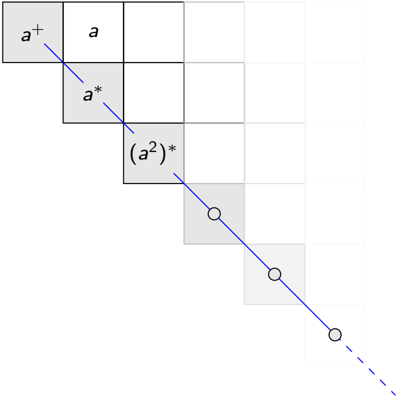


▸ Inverse as restriction

Varieties of restriction semigroups

- **M** is the variety of monoids: restriction semigroups with only one projection.
- Varieties of monoids play the role that varieties of groups play for inverse semigroup varieties.
- for example, the variety $\mathbf{SM} = \mathbf{SL} \vee \mathbf{M}$ of semilattices of monoids lies near the bottom of the lattice of varieties.

  These are the restriction semigroups that satisfy $x^+ = x^*$.
- If a variety does not consist of such semigroups, then it contains either B_0 or one of the semibicyclic semigroups B^+ or B^-.



The semibicyclic semigroup B^+

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Generalized Green's relations on restriction semigroups

- $\mathbb{R} = \{(a, b) : a^+ = b^+\}$
- $\mathbb{L} = \{(a, b) : a^* = b^*\}$
- $\mathbb{H} = \mathbb{L} \cap \mathbb{R}$
- $\mathbb{D} = \mathbb{L} \vee \mathbb{R}$
- $\mathbb{J}$: defined with respect to 'r-ideals'

These are the usual Green's relations on inverse semigroups (and the restrictions of those relations on their full subsemigroups).

Every $\mathbb{R}$-class and every $\mathbb{L}$-class contains a unique projection (a^+ and a^*, respectively).

In the 'York school', they are denoted $\widetilde{\mathcal{R}}_E$, etc.

$\mathbb{D}$-zigzags

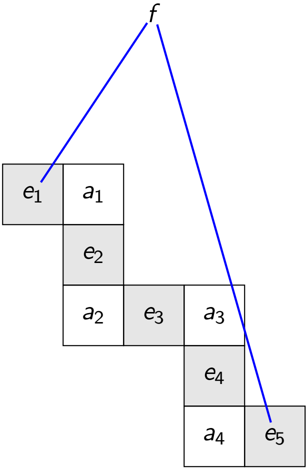
Projections e and f are $\mathbb{D}$-related if there is a a $\mathbb{D}$-zigzag between them. Here is a zigzag of length four that begins and ends in $\mathbb{L}$:

e_1	a_1			
	e_2			
	a_2	e_3	a_3	
			e_4	
			a_4	e_5

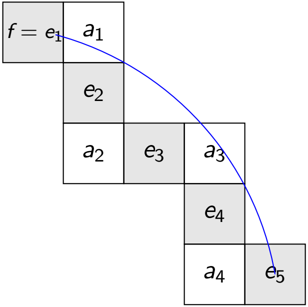
$\mathbb{D}$-majorization

A restriction semigroup is strict if it satisfies $\mathbb{D}$-majorization: no projection f is above distinct $\mathbb{D}$-related projections. This failure can occur in two essentially different manners, as illustrated in the next slide:

- a Λ_k-configuration.
- a Ψ_k-configuration.



A Λ_4-configuration. back



A Ψ_4-configuration. back

Strictness

Theorem

- *If S is a strict restriction semigroup, then it is a subdirect product of its 'principal r-factors', which are completely 0-r-simple semigroups.*
- *A completely 0-r-simple semigroup is a restriction semigroup with zero in which every projection is primitive and the nonzero elements form a single $\mathbb{D}$-class ('$\mathbb{D}$-0-simple').*
- *For example B_2 and B_0.*
- *Any completely 0-r-simple semigroup divides the direct product of a combinatorial Brandt semigroup and a monoid.*

The variety of strict restriction semigroups

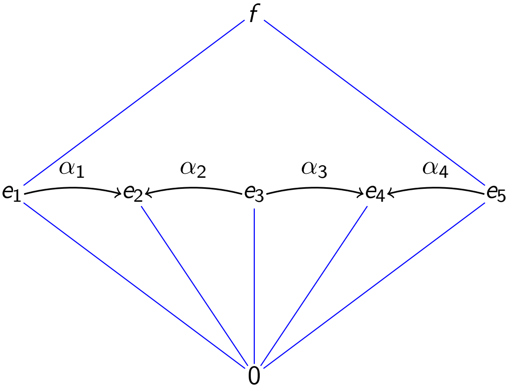
Theorem

The following are equivalent for a restriction semigroup:

- *S belongs to* **B**, *the variety generated by the Brandt semigroups;*
- *S is strict;*
- *S satisfies a certain sequence $(E_k)_{k \geq 1}$ of identities (which encapsulate $\mathbb{D}$-majorization).*

Λ_k

The semigroups Λ_k concretely manifest the failure of $\mathbb{D}$-majorization in the first scenario. Λ_k is a full subsemigroup of the Munn semigroup on the semilattice exemplified below.



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No finite set of identities will suffice

Theorem

The semigroup Λ_k satisfies all the identities (E_ℓ) for $\ell < k$ but does not satisfy (E_k).

Corollary

- *The variety* $\mathbf{B}$ *of restriction semigroups generated by the Brandt semigroups is not finitely based.*
- *The variety* $\mathcal{V}(B_2)$ *(consisting of the* $\mathbb{H}$*-combinatorial strict restriction semigroups) is not finitely based.*
- *The variety* $\mathcal{V}(B_0)$ *is not finitely based.*

Moving forward after my talk in Lisbon.

Hmmm... $\mathbb{D}$-majorization was characterized by lack of both Λ_k- and Ψ_k-configurations. What about the Ψ_k-configurations?

Concurrently, Volkov asked me if the semigroups Λ_k formed a 'critical series' for B_0, and I could easily see that the answer was 'yes'.

Failure of complete r-semisimplicity

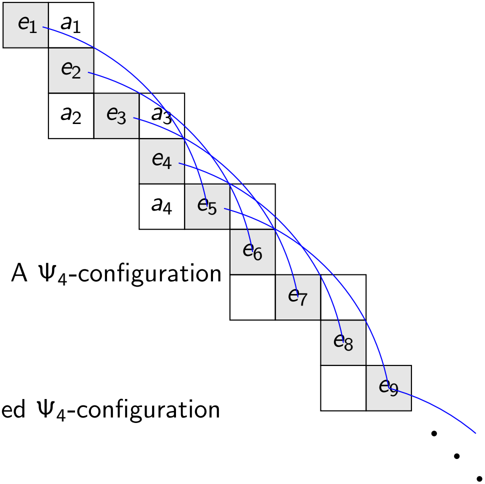
A restriction semigroup is completely r-semisimple if there do not exist distinct, comparable $\mathbb{D}$-related projections, that is, there are no Ψ_k-configurations.

Lemma

If a restriction semigroup S is not completely r-semisimple then it contains either

- *a Ψ_1-configuration, and so the semibicyclic semigroup B^+*
  B^+
- *a dual Ψ_1-configuration, and so B^-*
- *or a minimal Ψ_k-configuration for some* even k.

Ψ_k-configurations

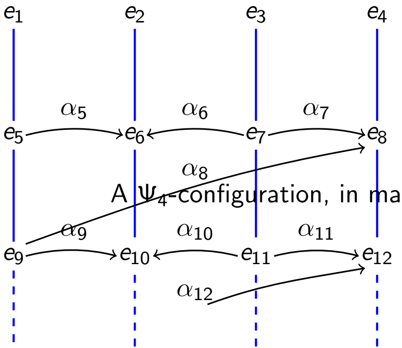


A Ψ_4-configuration

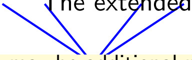
The extended Ψ_4-configuration

In mapping form

Lambda$_k$



A Ψ_4-configuration, in mapping form



The extended Ψ_4-configuration

The semigroup Ψ_4

Making Ψ_k concrete: its underlying semilattice is the 0-union of k ω-chains

There may be additional relations among the projections

The semigroups Ψ_k

Theorem

For any positive, even integer k, Ψ_k is an infinite, 0-$\mathbb{D}$-simple restriction semigroup that is generated, as such, by $\{\alpha_1, \ldots, \alpha_k\}$.

The elements α_i generate a null semigroup. In fact, 'almost all' products are zero.

The Ψ_k's form a critical series for B_0

Theorem

The semigroups Ψ_k comprise a series of critical restriction semigroups for B_0:

- $\Psi_k \notin \mathcal{V}(B_0)$
- Ψ_k *is* k*-generated*
- *each restriction subsemigroup of* Ψ_k *that is generated by fewer than* k *elements belongs to* $\mathcal{V}(B_0)$.

Critical series

Theorem

(Volkov) In general, if algebras A_k form a critical series for an algebra A, then any variety that contains A but no A_k's is nonfinitely based.

So any variety (of restriction semigroups) that contains B_0 but no Ψ_k is nonfinitely based.
Outline proof: if a variety $\mathbf{V}$ contains A but has a finite basis Σ of identities, then Σ involves words in no more than $k-1$ variables, say. But when evaluated in A_k, then each identity in Σ is actually evaluated in a subalgebra that belongs to $\mathcal{V}(A)$ and so to $\mathbf{V}$, and so is satisfied in A_k, contradicting $A_k \notin \mathbf{V}$.

Failure of complete r-semisimplicity in varietal terms

Theorem

- *If a restriction semigroup fails to be completely r-semisimple, it may not contain* B^+*,* B^- *or any* Ψ_k*. However,*
- *The variety it generates must contain one of these.*
- *And if that variety contains* B_0*, then it must contain some* Ψ_k*.*

Corollary

A variety of restriction semigroups contains B_0 *but no semigroups* Ψ_k *if and only if it all its members are completely r-semisimple.*

In comparison: a variety of restriction semigroups contains B_0 but no semigroups Λ_k if and only if it all its members are strict.

Theorem

- *Any variety of completely r-semisimple semigroups that contains B_0 is nonfinitely based.*
- *Any locally finite variety that contains B_0 is nonfinitely based, that is, B_0 is inherently nonfinitely based.*
  Proof. *No Ψ_k is locally finite.*
- *no finite restriction semigroup that is not simply a semilattice of monoids (i.e. doesn't satisfy $x^+ = x^*$) is finitely based.*

Theorem

The semigroups Λ_k also form a critical series for B_0, so B_0 is also not finitely based within the class of finite restriction semigroups.

Conclusion

- The four-element semigroup B_0 is inherently nonfinitely based.
- In fact the same is true for any finite restriction semigroup on which the two unary operations are not the same.
- B_0 and B_2 are finitely based as semigroups.
- B_2 is finitely based as an inverse semigroup.

References

- Peter R. Jones, On lattices of varieties of restriction semigroups, Semigroup Forum (2012), DOI:10.1007/s00233-012-9439-6 .
- Peter R. Jones, The semigroups B_2 and B_0 are inherently nonfinitely based, as restriction semigroups, submitted.
- M.V. Volkov, The finite basis problem for finite semigroups, Sci. Math. Jpn. 53 (2001), 171-199.