Expansions and covers

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Summary

This talk is partly based on an illuminating lecture given by Jon McCammond in Braga in 2003.

(1) Covers and expansions
(2) Mal'cev expansions
(3) Stabilisers
(4) Unitary semigroups

Removing singularities

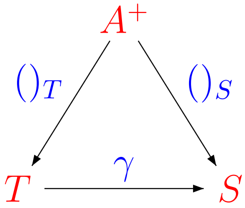
In mathematics, objects do not necessarily behave regularly and may sometimes have undesirable properties. A standard attempt to avoid such singularities is to replace defective objects by smoother ones.

The notion of cover in semigroup theory shares the same idea: removing singularities.

A-generated semigroups

An A-generated semigroup is a semigroup S together with a surjective morphism $u \to (u)_S$ from A^+ onto S. Then $(u)_S$ is called the value of u in S.

A morphism between two A-generated semigroups T and S is a surjective semigroup morphism $\gamma : T \to S$ such that the triangle below is commutative:



Covers

A cover associates to each semigroup S a semigroup $\widehat{S}$ and a surjective morphism $\pi_S : \widehat{S} \to S$.

Properties of the cover depend on the type of singularities to be removed. Properties of π are also sometimes required.

For instance, if S is an A-generated semigroup, the map

$$A^+ \xrightarrow{()_S} S$$

is the free cover of S. It gets rid of the relations between the generators.

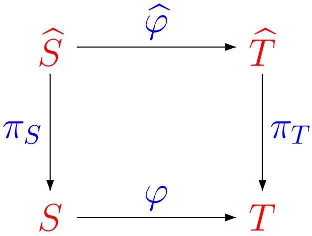
Expansions

An expansion is a functorial cover. It associates

(1) to each semigroup S a cover $\pi_S : \widehat{S} \to S$

(2) to each morphism $\varphi : S \to T$ a morphism $\widehat{\varphi} : \widehat{S} \to \widehat{T}$

such that the following diagram commutes:



Covers by ordered monoids

Theorem (Simon 75, Straubing-Thérien 85)

Every finite $\mathcal{J}$-trivial monoid is covered by a finite ordered monoid in which $x \leqslant 1$ for each element x.

Theorem (Henckell, Margolis, Pin, Rhodes)

Every finite monoid having at most one idempotent in each $\mathcal{R}$-class and in each $\mathcal{L}$-class is covered by a finite ordered monoid in which $e \leqslant 1$ for each idempotent e.

Varieties

A variety of semigroups is a class of semigroups closed under taking subsemigroups, quotient semigroups and direct products.

A semigroup is commutative iff it satisfies the identity $xy = yx$. A semigroup is idempotent iff it satisfies the identity $x^2 = x$.

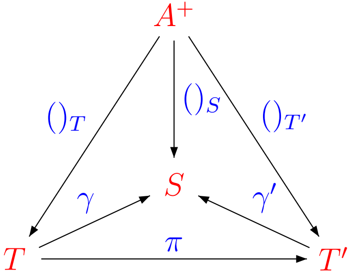
Let E be a set of identities. The variety of semigroups defined by E is the class $[\![E]\!]$ of all semigroups satisfying all identities of E.

Birkhoff's Theorem (1935). A class of semigroups is a variety iff it can be defined by a set of identities.

$\mathbf{V}$-extensions

Let $\mathbf{V}$ be a variety of semigroups. A semigroup morphism $\gamma : T \to S$ is a $\mathbf{V}$-extension of S if, for each idempotent $e \in S$, $\gamma^{-1}(e) \in \mathbf{V}$.

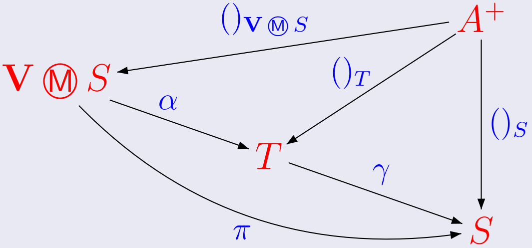
$\mathbf{V}$-extensions of an A-generated semigroup S form a category, whose morphisms are the morphisms $\pi : T \to T'$ such that this diagram is commutative:



The Mal'cev expansion as an initial object

Theorem (Universal property)

There is a $\mathbf{V}$*-extension of* S*, denoted* $\mathbf{V} Ⓜ S$*, such that for each* $\mathbf{V}$*-extension* $\gamma : T \to S$*, there is a morphism* $\alpha : \mathbf{V} Ⓜ S \to T$ *such that the following diagram commutes:*



Construction of the Mal'cev expansion (1)

Let S be an A-generated semigroup. A morphism $\sigma : B^+ \to A^+$ is said to be trivialized by S if there is an idempotent $e \in S$ such that $\big(\sigma(B^+)\big)_S = e$.

Note. It suffices to have $\big(\sigma(b)\big)_S = e$ for all $b \in B$.

Construction of the Mal'cev expansion (2)

Given a set E of identities defining $\mathbf{V}$, the Mal'cev expansion of S is be the semigroup $\mathbf{V} Ⓜ S$ with presentation

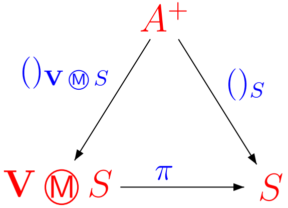
$$\langle A \mid \{ \sigma(u) = \sigma(v) \mid (u, v) \in B^+ \times B^+ \text{ is an identity of } E \text{ and } \sigma \text{ is trivialized by } S \}\rangle$$

Proposition

The definition of $\mathbf{V} Ⓜ S$ does not depend on the choice of the identities defining $\mathbf{V}$. Further it is functorial.

Construction of the Mal'cev expansion (3)

Each relator $\sigma(u) = \sigma(v)$ of the presentation of $\mathbf{V} \textcircled{M} S$ satisfies $(\sigma(u))_S = (\sigma(v))_S = \mathsf{e}$. Thus there is a unique surjective morphism $\pi : \mathbf{V} \textcircled{M} S \to S$ such that the following triangle commutes:



Theorem

The morphism π is a $\mathbf{V}$-extension of S.

Brown's Theorem

A semigroup is locally finite if all of its finitely generated subsemigroups are finite.

Theorem (Brown)

Let $\varphi : S \to T$ be a semigroup morphism. If T is locally finite and, for every idempotent $e \in T$, $\varphi^{-1}(e)$ is locally finite, then S is locally finite.

Locally finite varieties

A variety of semigroups $\mathbf{V}$ is locally finite if every finitely generated semigroup of $\mathbf{V}$ is finite.

Theorem

Let $\mathbf{V}$ be a locally finite variety, A a finite alphabet and S an A-generated semigroup. If S is finite, then $\mathbf{V} \circledM S$ is also finite.

Expansion by the trivial variety

Let $\mathbf{I}$ be the trivial variety of semigroups and let S be an A-semigroup. Then $\mathbf{I} Ⓜ S$ is the semigroup presented by $\langle A \mid \{u = v \mid (u)_S = (v)_S = (v^2)_S\}\rangle$.

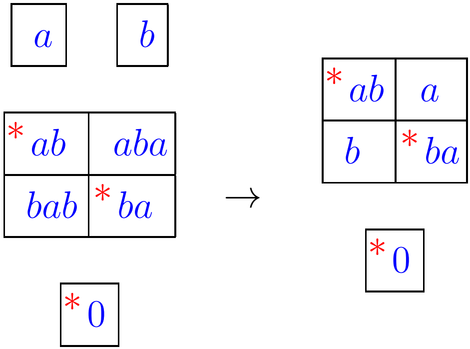
Proposition

Let S be a finite semigroup. Then the projection $\pi : \mathbf{I} Ⓜ S \to S$ is injective on regular elements: if x and y are regular elements of $\mathbf{I} Ⓜ S$, then $\pi(x) = \pi(y)$ implies $x = y$.

The $\mathbf{I}$-expansion of B_2

It is the semigroup presented by

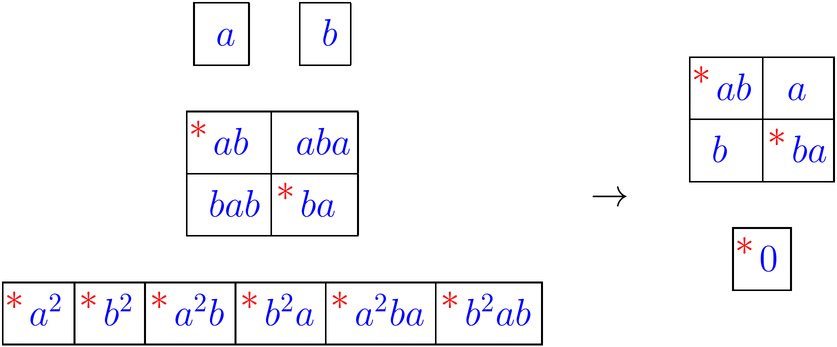
$\langle \{a, b\} \mid (ab)^2 = ab, (ba)^2 = ba, a^2 = b^2 = 0 \rangle$.



Mal'cev right-zero expansions

It is the semigroup presented by

$\langle A \mid \{vu = u \mid (v)_S = (u)_S = (u^2)_S\} \rangle$.



The Rhodes expansion

The left [right] Rhodes expansion of a semigroup S is an extension of S by right [left] zero semigroups.

Let S be an A-generated semigroup and let $(s_n, \ldots, s_0)$ be an $\leqslant_{\mathcal{L}}$-chain. The reduction $\rho(s_n, \ldots, s_0)$ is obtained from $(s_n, \ldots, s_0)$ by removing all the terms s_i such that $s_{i+1} \mathrel{\mathcal{L}} s_i$.

For instance, if $s_5 \mathrel{\mathcal{L}} s_4 \leqslant_{\mathcal{L}} s_3 \mathrel{\mathcal{L}} s_2 \mathrel{\mathcal{L}} s_1 \leqslant_{\mathcal{L}} s_0$, then $\rho(s_5, s_4, s_3, s_2, s_1, s_0) = (s_5, s_3, s_0)$.

The Rhodes expansion (2)

Denote by $L(S)$ the set of all $<_{\mathcal{L}}$-chains of S. Then the following operation makes $L(S)$ a semigroup:

$$(s_n, \ldots, s_0)(t_m, \ldots, t_0) = \rho(s_n t_m, s_{n-1} t_m, \ldots, s_0 t_m, t_m, \ldots, t_0)$$

The projection $\pi(s_n, \ldots, s_0) = s_n$ is a morphism from $L(S)$ onto S. Let $\widehat{\varphi} : A^+ \to L(S)$ be the morphism defined by $\widehat{\varphi}(a) = ((a)_S)$. The image $\widehat{S}^{\mathcal{L}} = \widehat{\varphi}(A^+)$ is the Rhodes expansion of S. Note that $()_S = \pi \circ \widehat{\varphi}$.

The Rhodes expansion of B_2

* (ab)	(a)
(b)	* (ba)

* $(0, a)$	* $(0, ab)$	* $(0, b)$	* $(0, ba)$

$\longrightarrow$

* ab	a
b	* ba

* 0

Properties of the Rhodes expansion $\widehat{S}^{\mathcal{L}}$

Let T be a semigroup. The right stabilizer of s is the semigroup $\{t \in T \mid st = s\}$

Proposition

(1) *An element* $(s_n, \ldots, s_0)$ *of* $\widehat{S}^{\mathcal{L}}$ *is idempotent iff* s_n *is idempotent in* S*.*

(2) *For each idempotent* e *of* S*,* $\pi^{-1}(e)$ *is a right zero semigroup.*

(3) *For each element* s *of* $\widehat{S}^{\mathcal{L}}$*, the right stabilizer of* s *is an* $\mathcal{R}$*-trivial semigroup.*

The Birget expansion

Obtained by iterating the left and right Rhodes expansion, alternatively: S, $\widehat{S}^{\mathcal{L}}$, $\widehat{\widehat{S}^{\mathcal{L}}}^{\mathcal{R}}$, $\widehat{\widehat{\widehat{S}^{\mathcal{L}}}^{\mathcal{R}}}^{\mathcal{L}}$, $\widehat{\widehat{\widehat{\widehat{S}^{\mathcal{L}}}^{\mathcal{R}}}^{\mathcal{L}}}^{\mathcal{R}}$

Theorem

If S is finite, this sequence ultimately stabilizes to a finite semigroup, the Birget expansion of S.
In the Birget expansion of a finite monoid, the $\leqslant_{\mathcal{R}}$-order on the $\mathcal{R}$-classes and the $\leqslant_{\mathcal{L}}$-order on the $\mathcal{L}$-classes form a tree.

Another expansion

Theorem (Le Saec, Pin, Weil 1991)

Every finite semigroup S is a quotient of a finite semigroup $\hat{S}$ in which the right stabilizer of any element is an $\mathcal{R}$-trivial band, that is, satisfies the identities $x^2 = x$ and $xyx = xy$.

T-covers

Let T be a submonoid of M.

T is dense in M if for each $u \in M$ there are elements $x, y \in M$ such that $xu, uy \in T$.

T is reflexive in M if $uv \in T$ implies $vu \in T$.

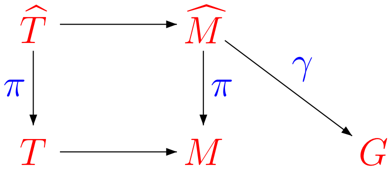
T is unitary in M if $u, uv \in T$ implies $v \in T$ and $u, vu \in T$ implies $v \in T$.

Proposition

T is a dense, reflexive and unitary subsemigroup of M iff there is a surjective morphism π from M onto a group G such that $T = \pi^{-1}(1)$.

T-covers

A T-cover of M is a monoid $\widehat{M}$ with a dense, reflexive, unitary submonoid $\widehat{T}$ of $\widehat{M}$ and a surjective morphism $\pi : \widehat{M} \to M$ onto M whose restriction to $\widehat{T}$ is an isomorphism from $\widehat{T}$ onto T.



E-unitary covers

An E-semigroup is a semigroup such that $E(S)$ is a subsemigroup.

An E-commutative semigroup is a semigroup in which the idempotents commute.

A monoid is E-dense [E-unitary] if $E(M)$ is a dense [unitary] submonoid of M.

A semigroup S is E-unitary [E-dense], if $E(S)$ is a unitary [dense] subsemigroup of S.

An orthodox semigroup is a regular E-semigroup.

E-commutative covers

Theorem (Fountain 1990)

(1) *Every E-commutative semigroup has an E-commutative unitary cover.*

(2) *Every E-commutative dense semigroup has an E-commutative unitary dense cover.*

(3) *Every inverse semigroup has an E-unitary inverse cover.*

See also McAlister, O'Carroll, Szendrei, Margolis and Pin for the inverse case.

E-unitary covers

Theorem (Almeida, Pin, Weil 1992)

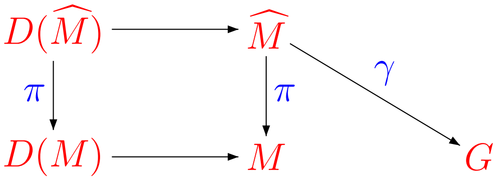
(1) *Every E-semigroup has an E-unitary cover.*

(2) *Every E-dense semigroup has an E-unitary dense cover.*

(3) *Every orthodox semigroup has an E-unitary orthodox cover.*

D-covers

Let $D(M)$ be the smallest submonoid of M closed under weak conjugation: if $x\bar{x}x = x$ and if $s \in D(M)$, then $xs\bar{x}, \bar{x}sx \in D(M)$.



D-covers

Theorem (Trotter 95)

Any regular monoid has a D-unitary regular cover.

Theorem (Fountain, Pin, Weil 2004)

Every E-dense monoid has a D-unitary E-dense cover.

The finite case

The following result is a consequence of the former Rhodes kernel conjecture, solved by Ash.

Theorem

Every finite monoid has a finite D-unitary cover. *Every finite E-semigroup has a finite E-unitary cover. If the semigroup is regular, the cover can be chosen regular as well.*