Regular and synchronizing transformation monoids

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I should add two remarks: João has helped me greatly with comments on a preliminary version of this talk; and he is pressing me to write a book on "Permutation groups for semigroupists"…

Permutation groups and transformation monoids

A permutation group is a subgroup of the symmetric group $\mathrm{Sym}(\Omega)$ on a set Ω. (Usually $\Omega = \{1, 2, \ldots, n\}$, and we write the symmetric group as S_n.) A transformation monoid is, analogously, a submonoid of the full transformation monoid $T(\Omega)$ on Ω (or T_n on $\{1, 2, \ldots, n\}$).

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Our knowledge of permutation groups has increased enormously since the Classification of Finite Simple Groups (CFSG) was announced in 1980. Can we bring this knowledge to bear on transformation semigroups?

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We cannot generate T_n with two elements, since we must include at least two permutations in any generating set. Moreover, permutations make up an exponentially small fraction of T_n. So we require many random elements to generate T_n with high probability.

Synchronization

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Conjecture

The probability that two random elements of T_n generate a synchronizing monoid tends to 1 *as* $n \to \infty$.

Here is some data produced by James Mitchell. The first row is the number n, the second is the number of such pairs of elements of T_n generating a synchronizing monoid, the third is the total number n^{2n} of pairs of elements of T_n, and the fourth is the second divided by the third.

3	4	5	6
549	51520	8063385	1871446896
729	65536	9765625	2176782336
0.7531	0.7861	0.8257	0.8597

These results were obtained using the Citrus and Orb packages for GAP.

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- Describe the maximal non-synchronizing submonoids of T_n;
- Use Inclusion-Exclusion to count the number of pairs of elements caught in one of these submonoids, and show that it is $o(n^{2n})$.

The first step has been achieved: the maximal non-sychronizing monoids have been characterised in terms of graphs, though there is still a gap between the necessary and sufficient conditions. Certainly, we do not understand these submonoids well enough to take the second step.

Synchronizing groups

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Araújo proved that G is *non-synchronizing* if and only if there is a non-trivial partition π of Ω and a subset S of Ω such that Sg is a transversal for π for all $g \in G$; equivalently, if S is a transversal for πg for all $g \in G$.

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(Here and subsequently, any structure M on Ω is trivial if $\mathrm{Aut}(M) = \mathrm{Sym}(\Omega)$. Thus, the trivial partitions are the partition into singletons and the partition with a single part.)
This was the result that started the study of synchronizing groups …

Primitivity

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Araújo observed that a synchronizing group is primitive, since if the non-trivial partition π is fixed by G, then we can take S to be any transversal for π to show that G is non-synchronizing. But the converse is not true.

Basic groups

The O’Nan–Scott Theorem has been crucial in the application of CFSG to permutation groups. It divides primitive groups into five classes, of which the last consists of almost simple groups, and in the other four we have good information about the action of the group.

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The first case consists of non-basic groups, those which preserve a Cartesian structure on Ω. More precisely, G is non-basic if there is a G-invariant bijection between Ω and the set of all l-tuples over an alphabet of size k, where $k, l > 1$, such that G preserves the Hamming metric on the set of tuples.

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Now a synchronizing group is basic. For if G is non-basic, take π to permute the set of tuples according to the value of the first coordinate, and S to be the set of constant tuples.

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The clique number $\omega(X)$ of the graph X is the size of the largest complete subgraph of X; and the chromatic number $\chi(X)$ of X is the smallest number of colours required for the vertices so that adjacent vertices have different colours. Clearly $\omega(X) \leq \chi(X)$.

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Theorem

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If such a graph exists, then the clique of size $\omega(X)$ is a transversal for any colouring with $\omega(X)$ colours, so G is non-synchronizing. The converse is not much more difficult.

This test is not computationally efficient: the number of non-trivial G-invariant graphs is $2^r - 2$, where r is the number of G-orbits on 2-element subsets of Ω; and finding clique number and chromatic number of a graph are NP-hard. (For graphs with a lot of symmetry, we can use the symmetry to speed up the computation, as is done in the GAP package Grape.)

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Note that a simple corollary of the theorem is that a 2-set transitive group is synchronizing.

An example

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The proof is quite complicated, using the Erdős–Ko–Rado theorem, the existence of Steiner triple systems, and Teirlinck's theorem on partitions into Steiner triple systems, as well as Lovász's Theorem on the chromatic number of the Kneser graph.

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The proof is quite complicated, using the Erdős–Ko–Rado theorem, the existence of Steiner triple systems, and Teirlinck's theorem on partitions into Steiner triple systems, as well as Lovász's Theorem on the chromatic number of the Kneser graph.

If we replace 3 by 4, we don't know the complete answer.

Non-synchronizing ranks

To quantify non-synchronization, I introduced the following idea. Let G be a permutation group on Ω. We say that the integer $r < n$ is a non-synchronizing rank of G if there exists a transformation f with rank r such that $\langle G, f \rangle$ is a non-synchronizing monoid. Let $\mathrm{NS}(G)$ be the set of non-synchronizing ranks of G.

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Thus, $\mathrm{NS}(G) = \emptyset$ if and only if G is synchronizing.

It is not hard to show that, if $2 \in \mathrm{NS}(G)$ or $n - 1 \in \mathrm{NS}(G)$, then G is imprimitive.

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- It is conjectured that, if G is primitive, then $NS(G)$ is *much* smaller, maybe only $O(\log n)$.

The maximal non-basic group $S_k \operatorname{wr} S_l$, acting on the Cartesian structure on $n = k^l$ points (the set of all l-tuples from $\{1, \ldots, k\}$) has the property that $k^i \in \mathrm{NS}(G)$ for $1 \leq i \leq l - 1$. Showing that this is the "worst case" involves some very intricate combinatorics and is not yet complete.

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> *Which permutation groups G have the property that, if f is any mapping of rank k, then f is regular in $\langle G, f\rangle$?*

We (João Araújo and I) are close to a solution to this question.

A stronger result

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Theorem

Suppose that G is a permutation group of degree n, and $k < n/2$. Then, with the exception of two groups with $n = 9$, $k = 4$, namely $\mathrm{AGL}(2,3)$ *and* $\mathrm{ASL}(2,3)$, *the following are equivalent:*

- *for every f of rank k, f is regular in $\langle G, f \rangle$;*
- *for every f of rank k, the semigroup $\langle G, f \rangle$ is regular.*

Translation to permutation groups

It is not hard to show that, if $fhf = f$ for some $h \in \langle G, f \rangle$, then we can choose h to be an element of G. So we can look in G for the answer to our question.

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Let f be a map on Ω. The image of f is what you think it is; the kernel of f is the partition of Ω into the sets $f^{-1}(a)$ for $a \in \mathrm{Im}(f)$.

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Let f be a map on Ω. The image of f is what you think it is; the kernel of f is the partition of Ω into the sets $f^{-1}(a)$ for $a \in \mathrm{Im}(f)$. Now if $h \in G$ satisfies $fhf = f$ then h maps $\mathrm{Im}(f)$ to a transversal for $\mathrm{Ker}(f)$.

We conclude that f is regular in $\langle G, f\rangle$ for every map f of rank k if and only if G has the following k-universal transversal property, or k-ut property for short:

> *Given any partition π with k parts, and every subset K of cardinality k, there exists $g \in G$ such that Kg is a transversal for π.*

So our question is:

> *Which permutation groups have the k-ut property, for given k?*

First reduction

Given natural numbers k, l with $k \leq l$, a permutation group G is (k, l)-set transitive if, given any sets K, L of cardinalities k, l respectively, there exists $g \in G$ such that $Kg \subseteq L$. If $k = l$, we just say k-set transitive.

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If G has the k-ut property, then it is $(k-1, k)$-set transitive. For suppose G has k-ut, and choose $K = \{a_1, \ldots, a_{k-1}\}$ and $L = \{b_1, \ldots, b_k\}$. Let π be the partition whose parts are the singletons of K and the whole of $\Omega \setminus K$. Then choose g mapping L to a transversal to π; then g^{-1} carries K into L.

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So a subquestion is:

Which permutation groups G are $(k-1, k)$-set transitive?

k-set transitivity

Investigating k-set transitivity, we see that it is equivalent to $(n-k)$-set transitivity, so we may assume that $k \leq n/2$. With this assumption, Livingstone and Wagner showed by an elegant argument that, for $k \geq 5$, k-set transitivity implies (and so is equivalent to) k-transitivity. (A permutation group is k-transitive if it acts transitively on the set of ordered k-tuples of distinct points.)

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Subsequently Kantor determined all the k-set transitive but not k-transitive permutation groups, for $k = 2, 3, 4$. He used results such as the Feit–Thompson theorem: groups of odd order are soluble.

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So these groups are well understood.

Classification of k-ut groups

Here are two of our results. For the first, note that $(k-1,k)$-set transitivity is equivalent to $(n-k,n-k+1)$-set transitivity, so we may assume that $k \leq n/2$.

Classification of k-ut groups

Here are two of our results. For the first, note that $(k-1,k)$-set transitivity is equivalent to $(n-k,n-k+1)$-set transitivity, so we may assume that $k \leq n/2$.

Theorem

Suppose that G is $(k-1,k)$-set transitive, with $k \leq n/2$. Then either G is $(k-1)$-set transitive, or G is one of five specific groups with $(n,k) = (5,2)$, $(7,3)$ or $(9,4)$.

The second result is maybe not quite a theorem. A permutation group G has the 2-ut property if and only if it is primitive. For given any 2-partition π and any 2-set S, if G is primitive then the graph with edge set $\{Sg : g \in G\}$ is connected, and so has at least one edge between parts of π.

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For the k-ut property with $k > 2$, we know that (with a few possible exceptions) G is $(k-1)$-set transitive, and hence "known". Of the known groups, some have the k-ut property and some do not; we have almost completely succeeded in deciding which is which. When complete, this would give a complete description of groups with the k-ut property.

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Note in particular that almost all groups with the k-ut property are $(k-1)$-set transitive, and hence have the l-ut property for all $l < k$.

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For $k = 4$, the group $G = M_{11}$ (with $n = 12$) is an exception, which fails to be 4-set transitive; any other such exception lies between $\mathrm{PSL}(2, q)$ and $\mathrm{P\Gamma L}(2, q)$, for prime powers q.

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- finite, if $k = 5$;
- infinite, if $k = 2$ or $k = 3$.

For $k = 4$, the group $G = M_{11}$ (with $n = 12$) is an exception, which fails to be 4-set transitive; any other such exception lies between $\mathrm{PSL}(2,q)$ and $\mathrm{P\Gamma L}(2,q)$, for prime powers q.
For $k = 3$, the groups for which we have not been able to resolve the question are the Suzuki groups.

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- A subgroup G of S_n is said to have the weak k-ut property if there exists a k-set S such that the orbit of S under G contains a transversal for all k-partitions. Such a set is called a G-universal transversal set. Classify the groups with the weak k-ut property; in addition, for each one of them, classify their G-universal transversal sets.


End


THANK YOU