Regular and synchronizing transformation monoids

Peter J. Cameron

NBSAN, York
23 November 2011
Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true.

Bertrand Russell
Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true.

Bertrand Russell

I don’t know much about semigroup theory (but João Araújo has encouraged me to think that the work we have done is of some importance). But at least I hope that what I tell you is true!
Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true.

Bertrand Russell

I don’t know much about semigroup theory (but João Araújo has encouraged me to think that the work we have done is of some importance). But at least I hope that what I tell you is true!
I should add two remarks: João has helped me greatly with comments on a preliminary version of this talk; and he is pressing me to write a book on “Permutation groups for semigroupists”…
A permutation group is a subgroup of the symmetric group Sym(Ω) on a set Ω. (Usually Ω = \{1, 2, \ldots, n\}, and we write the symmetric group as S_n.) A transformation monoid is, analogously, a submonoid of the full transformation monoid T(Ω) on Ω (or T_n on \{1, 2, \ldots, n\}).
A permutation group is a subgroup of the symmetric group $\text{Sym}(\Omega)$ on a set $\Omega$. (Usually $\Omega = \{1, 2, \ldots, n\}$, and we write the symmetric group as $S_n$.) A transformation monoid is, analogously, a submonoid of the full transformation monoid $T(\Omega)$ on $\Omega$ (or $T_n$ on $\{1, 2, \ldots, n\}$).

Our knowledge of permutation groups has increased enormously since the Classification of Finite Simple Groups (CFSG) was announced in 1980. Can we bring this knowledge to bear on transformation semigroups?
Dixon’s Theorem

**Theorem**

The probability that two random permutations of \( \{1, 2, \ldots, n\} \) generate the symmetric or alternating group tends to 1 as \( n \to \infty \).
Dixon’s Theorem

Theorem

The probability that two random permutations of \( \{1, 2, \ldots, n\} \) generate the symmetric or alternating group tends to 1 as \( n \to \infty \). We have to allow the alternating group since the probability that two random permutations are both even is \( 1/4 \).
Dixon’s Theorem

Theorem

The probability that two random permutations of \( \{1, 2, \ldots, n\} \) generate the symmetric or alternating group tends to 1 as \( n \to \infty \).

We have to allow the alternating group since the probability that two random permutations are both even is 1/4.

We cannot generate \( T_n \) with two elements, since we must include at least two permutations in any generating set. Moreover, permutations make up an exponentially small fraction of \( T_n \). So we require many random elements to generate \( T_n \) with high probability.
The **rank** of an element of $T_n$ is the cardinality of its image. A submonoid of $T_n$ is **synchronizing** if it contains an element of rank 1.
The **rank** of an element of $T_n$ is the cardinality of its image. A submonoid of $T_n$ is **synchronizing** if it contains an element of rank 1.

**Conjecture**

*The probability that two random elements of $T_n$ generate a synchronizing monoid tends to 1 as $n \to \infty$.***
Here is some data produced by James Mitchell. The first row is the number $n$, the second is the number of such pairs of elements of $T_n$ generating a synchronizing monoid, the third is the total number $n^{2n}$ of pairs of elements of $T_n$, and the fourth is the second divided by the third.

<table>
<thead>
<tr>
<th></th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>549</td>
<td>51520</td>
<td>8063385</td>
<td>1871446896</td>
</tr>
<tr>
<td></td>
<td>729</td>
<td>65536</td>
<td>9765625</td>
<td>2176782336</td>
</tr>
<tr>
<td></td>
<td>0.7531</td>
<td>0.7861</td>
<td>0.8257</td>
<td>0.8597</td>
</tr>
</tbody>
</table>

These results were obtained using the Citrus and Orb packages for GAP.
To prove this conjecture, following the proof of Dixon’s Theorem, there are two steps:

1. Describe the maximal non-synchronizing submonoids of $T_n$;
2. Use Inclusion-Exclusion to count the number of pairs of elements caught in one of these submonoids, and show that it is $o(n^2 n)$. 

The first step has been achieved: the maximal non-synchronizing monoids have been characterised in terms of graphs, though there is still a gap between the necessary and sufficient conditions. Certainly, we do not understand these submonoids well enough to take the second step.
To prove this conjecture, following the proof of Dixon’s Theorem, there are two steps:

- Describe the maximal non-synchronizing submonoids of $T_n$;

- Use Inclusion-Exclusion to count the number of pairs of elements caught in one of these submonoids, and show that it is $o(n^2n)$. The first step has been achieved: the maximal non-synchronizing monoids have been characterised in terms of graphs, though there is still a gap between the necessary and sufficient conditions. Certainly, we do not understand these submonoids well enough to take the second step.
To prove this conjecture, following the proof of Dixon’s Theorem, there are two steps:

- Describe the maximal non-synchronizing submonoids of $T_n$;
- Use Inclusion-Exclusion to count the number of pairs of elements caught in one of these submonoids, and show that it is $o(n^{2n})$. 
To prove this conjecture, following the proof of Dixon’s Theorem, there are two steps:

- Describe the maximal non-synchronizing submonoids of $T_n$;
- Use Inclusion-Exclusion to count the number of pairs of elements caught in one of these submonoids, and show that it is $o(n^{2n})$.

The first step has been achieved: the maximal non-synchronizing monoids have been characterised in terms of graphs, though there is still a gap between the necessary and sufficient conditions. Certainly, we do not understand these submonoids well enough to take the second step.
By abuse of language, we say that the permutation group $G$ on $\Omega = \{1, \ldots, n\}$ is **synchronizing** if $\langle G, f \rangle$ is a synchronizing monoid for any $f \in T_n \setminus S_n$. 

Araújo proved that $G$ is non-synchronizing if and only if there is a non-trivial partition $\pi$ of $\Omega$ and a subset $S$ of $\Omega$ such that $Sg$ is a transversal for $\pi$ for all $g \in G$; equivalently, if $S$ is a transversal for $\pi g$ for all $g \in G$. (Here and subsequently, any structure $M$ on $\Omega$ is trivial if $\text{Aut}(M) = \text{Sym}(\Omega)$. Thus, the trivial partitions are the partition into singletons and the partition with a single part.) 

This was the result that started the study of synchronizing groups...
Synchronizing groups

By abuse of language, we say that the permutation group $G$ on $\Omega = \{1, \ldots, n\}$ is **synchronizing** if $\langle G, f \rangle$ is a synchronizing monoid for any $f \in T_n \setminus S_n$.

Araújo proved that $G$ is **non-synchronizing** if and only if there is a non-trivial partition $\pi$ of $\Omega$ and a subset $S$ of $\Omega$ such that $Sg$ is a transversal for $\pi$ for all $g \in G$; equivalently, if $S$ is a transversal for $\pi g$ for all $g \in G$. 
By abuse of language, we say that the permutation group $G$ on $\Omega = \{1, \ldots, n\}$ is **synchronizing** if $\langle G, f \rangle$ is a synchronizing monoid for any $f \in T_n \setminus S_n$.

Araújo proved that $G$ is **non-synchronizing** if and only if there is a non-trivial partition $\pi$ of $\Omega$ and a subset $S$ of $\Omega$ such that $Sg$ is a transversal for $\pi$ for all $g \in G$; equivalently, if $S$ is a transversal for $\pi g$ for all $g \in G$.

(Here and subsequently, any structure $M$ on $\Omega$ is **trivial** if $\text{Aut}(M) = \text{Sym}(\Omega)$. Thus, the trivial partitions are the partition into singletons and the partition with a single part.)
By abuse of language, we say that the permutation group $G$ on $\Omega = \{1, \ldots, n\}$ is **synchronizing** if $\langle G, f \rangle$ is a synchronizing monoid for any $f \in T_n \setminus S_n$.

Araújo proved that $G$ is **non-synchronizing** if and only if there is a non-trivial partition $\pi$ of $\Omega$ and a subset $S$ of $\Omega$ such that $Sg$ is a transversal for $\pi$ for all $g \in G$; equivalently, if $S$ is a transversal for $\pi g$ for all $g \in G$.

(Here and subsequently, any structure $M$ on $\Omega$ is **trivial** if $\text{Aut}(M) = \text{Sym}(\Omega)$. Thus, the trivial partitions are the partition into singletons and the partition with a single part.)

This was the result that started the study of synchronizing groups …
A permutation group $G$ on $\Omega$ is primitive if $G$ preserves no non-trivial partition of $\Omega$. For $n > 2$, a primitive group is transitive (since otherwise the partition into an orbit and its complement is non-trivial).
A permutation group $G$ on $\Omega$ is primitive if $G$ preserves no non-trivial partition of $\Omega$. For $n > 2$, a primitive group is transitive (since otherwise the partition into an orbit and its complement is non-trivial).

Araújo observed that a synchronizing group is primitive, since if the non-trivial partition $\pi$ is fixed by $G$, then we can take $S$ to be any transversal for $\pi$ to show that $G$ is non-synchronizing.
Primitivity

A permutation group $G$ on $\Omega$ is **primitive** if $G$ preserves no non-trivial partition of $\Omega$. For $n > 2$, a primitive group is transitive (since otherwise the partition into an orbit and its complement is non-trivial).

Araújo observed that a synchronizing group is primitive, since if the non-trivial partition $\pi$ is fixed by $G$, then we can take $S$ to be any transversal for $\pi$ to show that $G$ is non-synchronizing. But the converse is not true.
Basic groups

The O’Nan–Scott Theorem has been crucial in the application of CFSG to permutation groups. It divides primitive groups into five classes, of which the last consists of almost simple groups, and in the other four we have good information about the action of the group.
Basic groups

The O’Nan–Scott Theorem has been crucial in the application of CFSG to permutation groups. It divides primitive groups into five classes, of which the last consists of almost simple groups, and in the other four we have good information about the action of the group.

The first case consists of non-basic groups, those which preserve a Cartesian structure on $\Omega$. More precisely, $G$ is non-basic if there is a $G$-invariant bijection between $\Omega$ and the set of all $l$-tuples over an alphabet of size $k$, where $k, l > 1$, such that $G$ preserves the Hamming metric on the set of tuples.
Basic groups

The O’Nan–Scott Theorem has been crucial in the application of CFSG to permutation groups. It divides primitive groups into five classes, of which the last consists of almost simple groups, and in the other four we have good information about the action of the group.

The first case consists of non-basic groups, those which preserve a Cartesian structure on $\Omega$. More precisely, $G$ is non-basic if there is a $G$-invariant bijection between $\Omega$ and the set of all $l$-tuples over an alphabet of size $k$, where $k, l > 1$, such that $G$ preserves the Hamming metric on the set of tuples.

Now a synchronizing group is basic. For if $G$ is non-basic, take $\pi$ to permute the set of tuples according to the value of the first coordinate, and $S$ to be the set of constant tuples.
Synchronization of $G$ can be detected in terms of $G$-invariant graphs, as follows.
Synchronization of $G$ can be detected in terms of $G$-invariant graphs, as follows.

The **clique number** $\omega(X)$ of the graph $X$ is the size of the largest complete subgraph of $X$; and the **chromatic number** $\chi(X)$ of $X$ is the smallest number of colours required for the vertices so that adjacent vertices have different colours. Clearly $\omega(X) \leq \chi(X)$. 

Theorem

The permutation group $G$ on $\Omega$ is non-synchronizing if and only if there is a non-trivial $G$-invariant graph $X$ on $\Omega$ with $\omega(X) = \chi(X)$.

If such a graph exists, then the clique of size $\omega(X)$ is a transversal for any colouring with $\omega(X)$ colours, so $G$ is non-synchronizing. The converse is not much more difficult.
Synchronization of $G$ can be detected in terms of $G$-invariant graphs, as follows.

The **clique number** $\omega(X)$ of the graph $X$ is the size of the largest complete subgraph of $X$; and the **chromatic number** $\chi(X)$ of $X$ is the smallest number of colours required for the vertices so that adjacent vertices have different colours. Clearly $\omega(X) \leq \chi(X)$.

**Theorem**

*The permutation group $G$ on $\Omega$ is non-synchronizing if and only if there is a non-trivial $G$-invariant graph $X$ on $\Omega$ with $\omega(X) = \chi(X)$.***
Synchronzation of $G$ can be detected in terms of $G$-invariant graphs, as follows.
The clique number $\omega(X)$ of the graph $X$ is the size of the largest complete subgraph of $X$; and the chromatic number $\chi(X)$ of $X$ is the smallest number of colours required for the vertices so that adjacent vertices have different colours. Clearly $\omega(X) \leq \chi(X)$.

**Theorem**
The permutation group $G$ on $\Omega$ is non-synchronizing if and only if there is a non-trivial $G$-invariant graph $X$ on $\Omega$ with $\omega(X) = \chi(X)$.

If such a graph exists, then the clique of size $\omega(X)$ is a transversal for any colouring with $\omega(X)$ colours, so $G$ is non-synchronizing. The converse is not much more difficult.
This test is not computationally efficient: the number of non-trivial $G$-invariant graphs is $2^r - 2$, where $r$ is the number of $G$-orbits on 2-element subsets of $\Omega$; and finding clique number and chromatic number of a graph are NP-hard. (For graphs with a lot of symmetry, we can use the symmetry to speed up the computation, as is done in the GAP package Grape.)
This test is not computationally efficient: the number of non-trivial $G$-invariant graphs is $2^r - 2$, where $r$ is the number of $G$-orbits on 2-element subsets of $\Omega$; and finding clique number and chromatic number of a graph are NP-hard. (For graphs with a lot of symmetry, we can use the symmetry to speed up the computation, as is done in the GAP package Grape.)

In this way, all primitive groups with degrees into the hundreds, and some with degrees in the thousands, have been tested, by Spiga and others.
This test is not computationally efficient: the number of non-trivial $G$-invariant graphs is $2^r - 2$, where $r$ is the number of $G$-orbits on 2-element subsets of $\Omega$; and finding clique number and chromatic number of a graph are NP-hard. (For graphs with a lot of symmetry, we can use the symmetry to speed up the computation, as is done in the GAP package Grape.) In this way, all primitive groups with degrees into the hundreds, and some with degrees in the thousands, have been tested, by Spiga and others. Note that a simple corollary of the theorem is that a 2-set transitive group is synchronizing.
An example

Let $G$ be the symmetric group of degree $n$ acting on the set $\Omega$ of 3-subsets of $\{1, \ldots, m\}$, with $n = \binom{m}{3}$. Then $G$ is primitive if $m \geq 7$. 

Theorem

$G$ is synchronizing if and only if $m$ is congruent to 2, 4, or 5 (mod 6) and $m \neq 8$. The proof is quite complicated, using the Erdős–Ko–Rado theorem, the existence of Steiner triple systems, and Teirlinck's theorem on partitions into Steiner triple systems, as well as Lovász's Theorem on the chromatic number of the Kneser graph. If we replace 3 by 4, we don't know the complete answer.
An example

Let $G$ be the symmetric group of degree $n$ acting on the set $\Omega$ of 3-subsets of $\{1, \ldots, m\}$, with $n = \binom{m}{3}$. Then $G$ is primitive if $m \geq 7$.

**Theorem**

$G$ is synchronizing if and only if $m$ is congruent to 2, 4 or 5 (mod 6) and $m \neq 8$. 
An example

Let $G$ be the symmetric group of degree $n$ acting on the set $\Omega$ of $3$-subsets of $\{1, \ldots, m\}$, with $n = \binom{m}{3}$. Then $G$ is primitive if $m \geq 7$.

**Theorem**

$G$ is synchronizing if and only if $m$ is congruent to $2, 4$ or $5 \pmod{6}$ and $m \neq 8$.

The proof is quite complicated, using the Erdős–Ko–Rado theorem, the existence of Steiner triple systems, and Teirlinck’s theorem on partitions into Steiner triple systems, as well as Lovász’s Theorem on the chromatic number of the Kneser graph.
An example

Let \( G \) be the symmetric group of degree \( n \) acting on the set \( \Omega \) of \( 3 \)-subsets of \( \{1, \ldots, m\} \), with \( n = \binom{m}{3} \). Then \( G \) is primitive if \( m \geq 7 \).

**Theorem**

\( G \) is synchronizing if and only if \( m \) is congruent to 2, 4 or 5 (mod 6) and \( m \neq 8 \).

The proof is quite complicated, using the Erdős–Ko–Rado theorem, the existence of Steiner triple systems, and Teirlinck’s theorem on partitions into Steiner triple systems, as well as Lovász’s Theorem on the chromatic number of the Kneser graph.

If we replace 3 by 4, we don’t know the complete answer.
Non-synchronizing ranks

To quantify non-synchronization, I introduced the following idea. Let $G$ be a permutation group on $\Omega$. We say that the integer $r < n$ is a non-synchronizing rank of $G$ if there exists a transformation $f$ with rank $r$ such that $\langle G, f \rangle$ is a non-synchronizing monoid. Let $\text{NS}(G)$ be the set of non-synchronizing ranks of $G$. 
Non-synchronizing ranks

To quantify non-synchronization, I introduced the following idea. Let $G$ be a permutation group on $\Omega$. We say that the integer $r < n$ is a non-synchronizing rank of $G$ if there exists a transformation $f$ with rank $r$ such that $\langle G, f \rangle$ is a non-synchronizing monoid. Let $\text{NS}(G)$ be the set of non-synchronizing ranks of $G$. Thus, $\text{NS}(G) = \emptyset$ if and only if $G$ is synchronizing.
To quantify non-synchronization, I introduced the following idea. Let $G$ be a permutation group on $\Omega$. We say that the integer $r < n$ is a non-synchronizing rank of $G$ if there exists a transformation $f$ with rank $r$ such that $\langle G, f \rangle$ is a non-synchronizing monoid. Let $NS(G)$ be the set of non-synchronizing ranks of $G$. Thus, $NS(G) = \emptyset$ if and only if $G$ is synchronizing. It is not hard to show that, if $2 \in NS(G)$ or $n - 1 \in NS(G)$, then $G$ is imprimitive.
The role of primitivity

From this point of view, it is once again primitivity that is the dividing line:
From this point of view, it is once again primitivity that is the dividing line:

- If $G$ is imprimitive, then $\text{NS}(G) \geq (3/4 - o(1))n$. 

The maximal non-basic group $S_k \wr S_l$, acting on the Cartesian structure on $n = kl$ points (the set of all $l$-tuples from $\{1, \ldots, k\}$) has the property that $k^i \in \text{NS}(G)$ for $1 \leq i \leq l - 1$. Showing that this is the “worst case” involves some very intricate combinatorics and is not yet complete.
The role of primitivity

From this point of view, it is once again primitivity that is the dividing line:

- If $G$ is imprimitive, then $\text{NS}(G) \geq (3/4 - o(1))n$.
- It is conjectured that, if $G$ is primitive, then $\text{NS}(G)$ is much smaller, maybe only $O(\log n)$. 
From this point of view, it is once again primitivity that is the dividing line:
  
  ▶ If $G$ is imprimitive, then $\text{NS}(G) \geq (3/4 - o(1))n$.
  
  ▶ It is conjectured that, if $G$ is primitive, then $\text{NS}(G)$ is much smaller, maybe only $O(\log n)$.

The maximal non-basic group $S_k \wr S_l$, acting on the Cartesian structure on $n = k^l$ points (the set of all $l$-tuples from $\{1, \ldots, k\}$) has the property that $k^i \in \text{NS}(G)$ for $1 \leq i \leq l - 1$. Showing that this is the “worst case” involves some very intricate combinatorics and is not yet complete.
Regularity

I probably don’t need to give this definition here ...
Regularity

I probably don’t need to give this definition here …

The element $a$ of a semigroup $S$ is regular in $S$ if there is an element $b$ of $S$ such that $aba = a$. A semigroup is regular if all its elements are.
I probably don’t need to give this definition here … The element $a$ of a semigroup $S$ is regular in $S$ if there is an element $b$ of $S$ such that $aba = a$. A semigroup is regular if all its elements are.

*Which* permutation groups $G$ *have the property that, if* $f$ *is any mapping of rank* $k$, *then* $f$ *is regular in* $\langle G, f \rangle$?
I probably don’t need to give this definition here … The element $a$ of a semigroup $S$ is regular in $S$ if there is an element $b$ of $S$ such that $aba = a$. A semigroup is regular if all its elements are.

Which permutation groups $G$ have the property that, if $f$ is any mapping of rank $k$, then $f$ is regular in $\langle G, f \rangle$?

We (João Araújo and I) are close to a solution to this question.
A stronger result

It turns out from our analysis that our condition on $G$ is almost equivalent to the statement that $\langle G, f \rangle$ is regular for all rank $k$ maps $f$. More precisely:
A stronger result

It turns out from our analysis that our condition on $G$ is almost equivalent to the statement that $\langle G, f \rangle$ is regular for all rank $k$ maps $f$. More precisely:

**Theorem**

*Suppose that $G$ is a permutation group of degree $n$, and $k < n/2$. Then, with the exception of two groups with $n = 9$, $k = 4$, namely $\text{AGL}(2, 3)$ and $\text{ASL}(2, 3)$, the following are equivalent:*

- for every $f$ of rank $k$, $f$ is regular in $\langle G, f \rangle$;
- for every $f$ of rank $k$, the semigroup $\langle G, f \rangle$ is regular.
Translation to permutation groups

It is not hard to show that, if $fhf = f$ for some $h \in \langle G, f \rangle$, then we can choose $h$ to be an element of $G$. So we can look in $G$ for the answer to our question.
It is not hard to show that, if $fhf = f$ for some $h \in \langle G, f \rangle$, then we can choose $h$ to be an element of $G$. So we can look in $G$ for the answer to our question.

Let $f$ be a map on $\Omega$. The **image** of $f$ is what you think it is; the **kernel** of $f$ is the partition of $\Omega$ into the sets $f^{-1}(a)$ for $a \in \text{Im}(f)$. 
Translation to permutation groups

It is not hard to show that, if $fhf = f$ for some $h \in \langle G, f \rangle$, then we can choose $h$ to be an element of $G$. So we can look in $G$ for the answer to our question.

Let $f$ be a map on $\Omega$. The **image** of $f$ is what you think it is; the **kernel** of $f$ is the partition of $\Omega$ into the sets $f^{-1}(a)$ for $a \in \text{Im}(f)$. Now if $h \in G$ satisfies $fhf = f$ then $h$ maps $\text{Im}(f)$ to a transversal for $\text{Ker}(f)$. 
We conclude that $f$ is regular in $\langle G, f \rangle$ for every map $f$ of rank $k$ if and only if $G$ has the following \textit{k-universal transversal property}, or \textit{k-ut property} for short:

\begin{quote}
Given any partition $\pi$ with $k$ parts, and every subset $K$ of cardinality $k$, there exists $g \in G$ such that $Kg$ is a transversal for $\pi$.
\end{quote}

So our question is:

\begin{quote}
Which permutation groups have the \textit{k-ut property}, for given $k$?
\end{quote}
First reduction

Given natural numbers $k, l$ with $k \leq l$, a permutation group $G$ is \((k, l)\)-set transitive if, given any sets $K, L$ of cardinalities $k, l$ respectively, there exists $g \in G$ such that $Kg \subseteq L$. If $k = l$, we just say $k$-set transitive.
First reduction

Given natural numbers $k, l$ with $k \leq l$, a permutation group $G$ is \((k, l)\)-set transitive if, given any sets $K, L$ of cardinalities $k, l$ respectively, there exists $g \in G$ such that $Kg \subseteq L$. If $k = l$, we just say $k$-set transitive.

It is common to say “homogeneous” rather than “set-transitive”; I have avoided this since with another hat on I work on homogeneous structures, and the two meanings of the term do not sit well together.
Given natural numbers \( k, l \) with \( k \leq l \), a permutation group \( G \) is \((k, l)\)-set transitive if, given any sets \( K, L \) of cardinalities \( k, l \) respectively, there exists \( g \in G \) such that \( Kg \subseteq L \). If \( k = l \), we just say \( k \)-set transitive.

It is common to say “homogeneous” rather than “set-transitive”; I have avoided this since with another hat on I work on homogeneous structures, and the two meanings of the term do not sit well together.

If \( G \) has the \( k \)-ut property, then it is \((k − 1, k)\)-set transitive. For suppose \( G \) has \( k \)-ut, and choose \( K = \{a_1, \ldots, a_{k−1}\} \) and \( L = \{b_1, \ldots, b_k\} \). Let \( \pi \) be the partition whose parts are the singletons of \( K \) and the whole of \( \Omega \setminus K \). Then choose \( g \) mapping \( L \) to a transversal to \( \pi \); then \( g^{-1} \) carries \( K \) into \( L \).
Given natural numbers $k, l$ with $k \leq l$, a permutation group $G$ is \textbf{$(k, l)$-set transitive} if, given any sets $K, L$ of cardinalities $k, l$ respectively, there exists $g \in G$ such that $Kg \subseteq L$. If $k = l$, we just say \textbf{$k$-set transitive}.

It is common to say "homogeneous" rather than "set-transitive"; I have avoided this since with another hat on I work on homogeneous structures, and the two meanings of the term do not sit well together.

If $G$ has the $k$-ut property, then it is $(k - 1, k)$-set transitive. For suppose $G$ has $k$-ut, and choose $K = \{a_1, \ldots, a_{k-1}\}$ and $L = \{b_1, \ldots, b_k\}$. Let $\pi$ be the partition whose parts are the singletons of $K$ and the whole of $\Omega \setminus K$. Then choose $g$ mapping $L$ to a transversal to $\pi$; then $g^{-1}$ carries $K$ into $L$.

So a subquestion is:

\textit{Which permutation groups $G$ are $(k - 1, k)$-set transitive?}
Investigating $k$-set transitivity, we see that it is equivalent to $(n - k)$-set transitivity, so we may assume that $k \leq n/2$. With this assumption, Livingstone and Wagner showed by an elegant argument that, for $k \geq 5$, $k$-set transitivity implies (and so is equivalent to) $k$-transitivity. (A permutation group is $k$-transitive if it acts transitively on the set of ordered $k$-tuples of distinct points.)
Investigating $k$-set transitivity, we see that it is equivalent to $(n - k)$-set transitivity, so we may assume that $k \leq n/2$. With this assumption, Livingstone and Wagner showed by an elegant argument that, for $k \geq 5$, $k$-set transitivity implies (and so is equivalent to) $k$-transitivity. (A permutation group is $k$-transitive if it acts transitively on the set of ordered $k$-tuples of distinct points.) Subsequently Kantor determined all the $k$-set transitive but not $k$-transitive permutation groups, for $k = 2, 3, 4$. He used results such as the Feit–Thompson theorem: groups of odd order are soluble.
Investigating $k$-set transitivity, we see that it is equivalent to $(n - k)$-set transitivity, so we may assume that $k \leq n/2$. With this assumption, Livingstone and Wagner showed by an elegant argument that, for $k \geq 5$, $k$-set transitivity implies (and so is equivalent to) $k$-transitivity. (A permutation group is $k$-transitive if it acts transitively on the set of ordered $k$-tuples of distinct points.) Subsequently Kantor determined all the $k$-set transitive but not $k$-transitive permutation groups, for $k = 2, 3, 4$. He used results such as the Feit–Thompson theorem: groups of odd order are soluble.

By the time CFSG was announced in 1980, it was known that the classification of $k$-transitive groups for $k \geq 2$ would follow from it: in particular, the only $k$-transitive groups for $k \geq 6$ are the symmetric and alternating groups.
Investigating $k$-set transitivity, we see that it is equivalent to $(n - k)$-set transitivity, so we may assume that $k \leq n/2$. With this assumption, Livingstone and Wagner showed by an elegant argument that, for $k \geq 5$, $k$-set transitivity implies (and so is equivalent to) $k$-transitivity. (A permutation group is $k$-transitive if it acts transitively on the set of ordered $k$-tuples of distinct points.) Subsequently Kantor determined all the $k$-set transitive but not $k$-transitive permutation groups, for $k = 2, 3, 4$. He used results such as the Feit–Thompson theorem: groups of odd order are soluble. By the time CFSG was announced in 1980, it was known that the classification of $k$-transitive groups for $k \geq 2$ would follow from it: in particular, the only $k$-transitive groups for $k \geq 6$ are the symmetric and alternating groups. So these groups are well understood.
Classification of $k$-ut groups

Here are two of our results. For the first, note that $(k - 1, k)$-set transitivity is equivalent to $(n - k, n - k + 1)$-set transitivity, so we may assume that $k \leq n/2$. 
Classification of $k$-ut groups

Here are two of our results. For the first, note that $(k−1, k)$-set transitivity is equivalent to $(n − k, n − k + 1)$-set transitivity, so we may assume that $k \leq n/2$.

**Theorem**

*Suppose that $G$ is $(k − 1, k)$-set transitive, with $k \leq n/2$. Then either $G$ is $(k − 1)$-set transitive, or $G$ is one of five specific groups with $(n, k) = (5, 2), (7, 3)$ or $(9, 4)$.***
The second result is maybe not quite a theorem. A permutation group $G$ has the 2-ut property if and only if it is primitive. For given any 2-partition $\pi$ and any 2-set $S$, if $G$ is primitive then the graph with edge set $\{Sg : g \in G\}$ is connected, and so has at least one edge between parts of $\pi$. 
The second result is maybe not quite a theorem. A permutation group \( G \) has the 2-ut property if and only if it is primitive. For given any 2-partition \( \pi \) and any 2-set \( S \), if \( G \) is primitive then the graph with edge set \( \{ Sg : g \in G \} \) is connected, and so has at least one edge between parts of \( \pi \).

For the \( k \)-ut property with \( k > 2 \), we know that (with a few possible exceptions) \( G \) is \((k - 1)\)-set transitive, and hence “known”. Of the known groups, some have the \( k \)-ut property and some do not; we have almost completely succeeded in deciding which is which. When complete, this would give a complete description of groups with the \( k \)-ut property.
The second result is maybe not quite a theorem. A permutation group $G$ has the 2-ut property if and only if it is primitive. For given any 2-partition $\pi$ and any 2-set $S$, if $G$ is primitive then the graph with edge set $\{Sg : g \in G\}$ is connected, and so has at least one edge between parts of $\pi$.

For the $k$-ut property with $k > 2$, we know that (with a few possible exceptions) $G$ is $(k - 1)$-set transitive, and hence “known”. Of the known groups, some have the $k$-ut property and some do not; we have almost completely succeeded in deciding which is which. When complete, this would give a complete description of groups with the $k$-ut property. Note in particular that almost all groups with the $k$-ut property are $(k - 1)$-set transitive, and hence have the $l$-ut property for all $l < k$. 
It turns out that, for fixed \( k \), the class of groups which have the \( k \)-ut property but are not symmetric or alternating, is

- empty, if \( 6 \leq k \leq n/2 \);
A comment

It turns out that, for fixed $k$, the class of groups which have the $k$-ut property but are not symmetric or alternating, is

- empty, if $6 \leq k \leq n/2$;
- finite, if $k = 5$;
- infinite, if $k = 2$ or $k = 3$.

For $k = 4$, the group $G = M_{11}$ (with $n = 12$) is an exception, which fails to be 4-set transitive; any other such exception lies between $PSL(2, q)$ and $P\Gamma L(2, q)$, for prime powers $q$.

For $k = 3$, the groups for which we have not been able to resolve the question are the Suzuki groups.
A comment

It turns out that, for fixed $k$, the class of groups which have the $k$-ut property but are not symmetric or alternating, is

- empty, if $6 \leq k \leq n/2$;
- finite, if $k = 5$;
- infinite, if $k = 2$ or $k = 3$. 
It turns out that, for fixed $k$, the class of groups which have the $k$-ut property but are not symmetric or alternating, is

- empty, if $6 \leq k \leq n/2$;
- finite, if $k = 5$;
- infinite, if $k = 2$ or $k = 3$.

For $k = 4$, the group $G = M_{11}$ (with $n = 12$) is an exception, which fails to be 4-set transitive; any other such exception lies between $\text{PSL}(2, q)$ and $\text{PGL}(2, q)$, for prime powers $q$. 
It turns out that, for fixed $k$, the class of groups which have the $k$-ut property but are not symmetric or alternating, is

- empty, if $6 \leq k \leq n/2$;
- finite, if $k = 5$;
- infinite, if $k = 2$ or $k = 3$.

For $k = 4$, the group $G = M_{11}$ (with $n = 12$) is an exception, which fails to be 4-set transitive; any other such exception lies between $\text{PSL}(2, q)$ and $\text{PGL}(2, q)$, for prime powers $q$. For $k = 3$, the groups for which we have not been able to resolve the question are the Suzuki groups.
Open problems

Among many, I just mention a couple here.

▶ Prove analogues of these results in other cases, such as submonoids of the monoid of endomorphisms of a finite-dimensional vector space, or more generally of an independence algebra of finite rank.

▶ Classify the pairs \((G, f)\), where \(G \leq S_n\) and \(f \in T_n \setminus S_n\), for which \(\langle G, f \rangle\) is regular. (McAlister proved that this holds if \(G\) is any permutation group and \(f\) an idempotent of rank \(n - 1\)).

▶ A subgroup \(G\) of \(S_n\) is said to have the weak \(k\)-ut property if there exists a \(k\)-set \(S\) such that the orbit of \(S\) under \(G\) contains a transversal for all \(k\)-partitions. Such a set is called a \(G\)-universal transversal set. Classify the groups with the weak \(k\)-ut property; in addition, for each one of them, classify their \(G\)-universal transversal sets.
Open problems

Among many, I just mention a couple here.

- Prove analogues of these results in other cases, such as submonoids of the monoid of endomorphisms of a finite-dimensional vector space, or more generally of an independence algebra of finite rank.
Open problems

Among many, I just mention a couple here.

- Prove analogues of these results in other cases, such as submonoids of the monoid of endomorphisms of a finite-dimensional vector space, or more generally of an independence algebra of finite rank.

- Classify the pairs \((G,f)\), where \(G \leq S_n\) and \(f \in T_n \setminus S_n\), for which \(\langle G,f \rangle\) is regular. (McAlister proved that this holds if \(G\) is any permutation group and \(f\) an idempotent of rank \(n - 1\).)
Open problems

Among many, I just mention a couple here.

- Prove analogues of these results in other cases, such as submonoids of the monoid of endomorphisms of a finite-dimensional vector space, or more generally of an independence algebra of finite rank.

- Classify the pairs \((G,f)\), where \(G \leq S_n\) and \(f \in T_n \setminus S_n\), for which \(\langle G,f \rangle\) is regular. (McAlister proved that this holds if \(G\) is any permutation group and \(f\) an idempotent of rank \(n - 1\).)

- A subgroup \(G\) of \(S_n\) is said to have the weak \(k\)-ut property if there exists a \(k\)-set \(S\) such that the orbit of \(S\) under \(G\) contains a transversal for all \(k\)-partitions. Such a set is called a \(G\)-universal transversal set. Classify the groups with the weak \(k\)-ut property; in addition, for each one of them, classify their \(G\)-universal transversal sets.