Direct and subdirect products of groups, semigroups and algebras

$$
\begin{tikzcd}
P \arrow[r, "f_T"] \arrow[d, "f_T"] \arrow[dr, dashed, "f"] & T \\
S & S \times T \arrow[l, "\pi_S"] \arrow[u, "\pi_T"']
\end{tikzcd}
$$

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Context and acknowledgements

These lecture notes are based on the five part digital lecture series, entitled "Direct and subdirect products of groups, semigroups and algebras" given to the North British Semigroups & their Applications Network (NBSAN) in September 2020. They are provided to accompany the digitally recorded videos of the lectures found on the London Mathematical Society's (LMS) YouTube channel. These lectures were kindly financially supported by the LMS, and further supported by the organisers of the NBSAN.

Motivation and content

This five part lecture series is based on the topics of "Direct and subdirect products in groups, semigroups and algebras". The intended audience for this series consists of postgraduates entering a PhD in any form of combinatorial algebra, though indeed anyone with a familiarity from undergraduate courses in this area may also benefit, particularly postgraduate students at later stages.

In summary, the aim of the lecture series is introduce the algebraic construction of the direct product and the related notion of subdirect products, in the settings of group theory (which should already be familiar from undergraduate courses), semigroup theory and more widely in universal algebra. Starting with the familiar case of groups, we will discuss both the abstract motivation and uses of the direct product construction in reference to direct decomposability. We will then emphasise abstraction of this construction to the related areas of semigroup theory and universal algebra. Some of the shortcomings of the direct product construction will then be used to motivate the theory of subdirect products and subdirect irreducibility. We will then discuss the effects of these constructions on structural properties in all of these algebraic structures in the latter parts of the series. Further, we introduce some classic and recent research questions concerning direct and subdirect products, with the intention of highlighting some active research topics; providing potential research directions for postgraduates; and promoting the area of semigroup theory and research interests represented within NBSAN.

Chapter 1

Introduction to direct products

In the study of algebraic structures, a common theme of interest is representing a given structure in terms of known and well studied simpler structures. Such representations can be important when determining the properties of a given structure, by knowing the properties of the simpler structures that can represent it.

For example, in finite group theory, the finite *simple* groups are considered as the “building blocks” that make up a finite group, and we want to know in what ways that finite group be built or represented by them. There are of course a number of ways to construct another group given two or more simple groups, such as through wreath products, semidirect products, and others. The direct product, however, is probably the simplest such construction method.

This example from group theory is still a motivator in the theories of more general algebraic structures. In general, the construction of the direct product of two algebraic structures is a relatively basic and well-behaved operation. It is often true, particularly in the class of groups, that algebraic properties such as finite generation, finite presentation, and solubility (amongst others) are preserved by direct products. Moreover, the converse is also true; each factor in a direct product often inherits such properties.

Of course, direct products are not restricted to the class of groups, and comparing and contrasting the effect of the direct product on generalisations of these properties between different classes of algebras is of interest. However, we will begin by defining the direct product construction in the context of group theory, and examine some of its properties.

1.1 Direct products of groups

To avoid any ambiguity in notation or meaning, we will start by recalling some basic preliminaries from set theory and group theory, which are familiar from undergraduate courses.

Definition 1.1.1. The *Cartesian product* of two sets S and T is the set of pairs given by

$$S \times T = \{(s,t) : s \in S, t \in T\}. \qquad \blacksquare$$

Definition 1.1.2. A *group* is a set G together with a binary operation $\cdot : G \times G \to G$, satisfying the following axioms:

- (Associativity) $\forall f, g, h \in G, \quad (f \cdot g) \cdot h = f \cdot (g \cdot h)$;
- (Identity) $\exists\, 1 \in G$ such that $\forall g \in G, \quad 1 \cdot g = g \cdot 1 = g$;
- (Inverses) $\forall g \in G$, $\exists g^{-1} \in G$ such that $g \cdot g^{-1} = g^{-1} \cdot g = 1$.

The operation $\cdot$ will be called the *multiplication* of G, and as usual will often be omitted notationally when the context is clear. $\blacksquare$

As another standard convention, we will often refer to the group G by its set only. That is, we won't necessarily pair G with its operation $\cdot$ explicitly every time, writing just G to mean $(G, \cdot)$.

Definition 1.1.3. For two groups G and H with operations $\cdot_G$ and $\cdot_H$ respectively, the *direct product of G and H* is the Cartesian product $G \times H$, together with the binary operation $\cdot : (G \times H) \times (G \times H) \to (G \times H)$ defined by

$$(g, h) \cdot (g', h') = (g \cdot_G g', h \cdot_H h'). \qquad \blacksquare$$

Where a binary operation is defined, a structure is sure to follow. The next proposition indeed verifies that the direct product of two groups is a group.

Proposition 1.1.4. *The direct product of groups G and H as defined in Definition* 1.1.3 *is a group.*

Proof. The full proof is left as an exercise to the reader, but associativity follows from associativity of $\cdot_G$ and $\cdot_H$, the identity of the group is $(1_G, 1_H)$, and the inverse of (g, h) is (g^{-1}, h^{-1}). $\square$

Examples 1.1.5.

- The direct product of $\mathbb{Z}_2 = \{0, 1\}$ (the finite cyclic group of order 2) with itself is a group, which is isomorphic to the Klein four-group K_4 with multiplication table

$$\begin{array}{c|cccc} \cdot & 1 & i & j & k \\ \hline 1 & 1 & i & j & k \\ i & i & 1 & k & j \\ j & j & k & 1 & i \\ k & k & j & i & 1 \end{array}$$

  One such isomorphism is $\varphi : \mathbb{Z}_2 \times \mathbb{Z}_2 \to K_4$ given by $\varphi(0,0) = 1$, $\varphi(1,0) = i$, $\varphi(0,1) = j$ and $\varphi(1,1) = k$.

- The vector space $\mathbb{R}^2$ is a direct product of $\mathbb{R}$ with itself, when viewed as a group.

- Every group G can be realised as a direct product of groups, as $G \cong G \times \{1\}$ (where $\{1\}$ is the trivial group). $\triangle$

Note: Of course, there was no particular reason to choose only two groups for our definition of direct products: we can make direct products of three, four, or any finite number of n groups in the natural way. The product of two n-tuples is another n-tuple, whose i-th coordinate is the product of the two given i-th coordinates.

In fact, given any indexed family of groups $\{G_i\}_{i\in\mathcal{I}}$ with indexing set $\mathcal{I}$, we can make a direct product of them in the following manner.

Definition 1.1.6. For an indexed family of sets $\{G_i\}_{i\in\mathcal{I}}$, the *Cartesian product* $\prod_{i\in\mathcal{I}} G_i$ is defined by

$$\prod_{i\in\mathcal{I}} G_i := \left\{ f : \mathcal{I} \to \bigcup_{i\in\mathcal{I}} G_i : \text{ for all } i \in \mathcal{I}, f(i) \in G_i \right\}.$$

The *direct product* of the family of groups $\{G_i\}_{i\in\mathcal{I}}$ is the Cartesian product $G = \prod_{i\in\mathcal{I}} G_i$ with operation $\cdot : G \times G \to G$, for which the product of two functions $f, g \in G$ is the function $f \cdot g : \mathcal{I} \to \bigcup_{i\in\mathcal{I}} G_i$ given by

$$(f \cdot g)(i) = f(i) \cdot_{G_i} g(i). \qquad \blacksquare$$

Again, let's see that this definition of the direct product gives us a group.

Proposition 1.1.7. *The direct product of a family of groups $G = \{G_i\}_{i\in\mathcal{I}}$ as defined in Definition* 1.1.6 *is a group.*

Proof. Let $f, g, h \in G$. By definition, $f \cdot g$ is a function from $\mathcal{I}$ to $\bigcup_{i\in\mathcal{I}} G_i$, such that for all $i \in \mathcal{I}$,

$$(f \cdot g)(i) = f(i) \cdot_{G_i} g(i) \in G_i$$

by closure of the operation on G_i. Hence $\cdot : G \times G \to G$ is a binary operation.

To show associativity, we have

$$\begin{aligned}
[(f \cdot g) \cdot h](i) &= (f \cdot g)(i) \cdot_{G_i} h(i) \\
&= [f(i) \cdot_{G_i} g(i)] \cdot_{G_i} h(i) \\
&= f(i) \cdot_{G_i} [g(i) \cdot_{G_i} h(i)] \quad \text{(By associativity in } G_i) \\
&= f(i) \cdot_{G_i} (g \cdot h)(i) \\
&= [f \cdot (g \cdot h)](i),
\end{aligned}$$

and hence we have shown that $(f \cdot g) \cdot h = f \cdot (g \cdot h)$. The identity of G is the function $e : \mathcal{I} \to \bigcup_{i\in\mathcal{I}} G_i$ given by $e(i) = 1_{G_i}$ (the identity of G_i), as

$$(f \cdot e)(i) = f(i) \cdot_{G_i} e(i) = f(i) \cdot_{G_i} 1_{G_i} = f(i)$$

and

$$(e \cdot f)(i) = e(i) \cdot_{G_i} f(i) = 1_{G_i} \cdot_{G_i} f(i) = f(i),$$

hence $f \cdot e = f = e \cdot f$. Finally, the inverse of f is the function $f^{-1} : \mathcal{I} \to \bigcup_{i\in\mathcal{I}} G_i$ given by $f^{-1}(i) = (f(i))^{-1}$, as

$$(f^{-1} \cdot f)(i) = f^{-1}(i) \cdot_{G_i} f(i) = (f(i))^{-1} \cdot_{G_i} f(i) = 1_{G_i} = e(i)$$

and

$$(f \cdot f^{-1})(i) = f(i) \cdot_{G_i} f^{-1}(i) = f(i) \cdot_{G_i} (f(i))^{-1} = 1_{G_i} = e(i),$$

hence $f \cdot f^{-1} = f^{-1} \cdot f = e$. Note that the function f^{-1} is **not** necessarily the inverse function of f. □

When $\mathcal{I}$ is finite with $|\mathcal{I}| = n$, there is a natural isomorphism between $\prod_{i\in\mathcal{I}} G_i$ and $G_1 \times G_2 \times \cdots \times G_n$. See the following exercise for the $n = 2$ case.

Exercise 1.1.8. Let G and H be groups, $\mathcal{I} = \{1, 2\}$, $G_1 = G$, $G_2 = H$. Show that the map

$$\varphi : \prod_{i\in\{1,2\}} G_i \to G \times H,\ f \mapsto (f(1), f(2))$$

is an isomorphism of groups. Hence given groups $G_1, G_2, \ldots, G_n$, generalise this map to find an isomorphism between $\prod_{i\in\mathcal{I}} G_i$ and $G_1 \times G_2 \times \cdots \times G_n$.

Note: For the rest of this series, by "the direct product" of G and H, we will mean $G \times H$ rather than Definition 1.1.6.

1.2 Direct products of semigroups and monoids

Groups are only one example of an algebraic structure with a direct product construction. There are many others, but we will first focus on two classes of algebras that generalise groups. These are the classes of *semigroups* and *monoids*.

Definition 1.2.1. A *semigroup* is a pair $(S, \cdot)$, where S is a non-empty set, and $\cdot : S \times S \to S$ is a binary operation that is also *associative*, meaning for all $s, t, u \in S$,

$$s \cdot (t \cdot u) = (s \cdot t) \cdot u. \qquad \blacksquare$$

Where the context of the operation is clearly understood, we will speak of the semigroup S only by referring to its set. We will also omit the symbol $\cdot$, and instead write st for $s \cdot t$, which we will call the *product* of s and t. We will further talk about the operation as the *multiplication* of S. The multiplicative notation allows us to consider *powers* of elements s^n for $n \in \mathbb{N}$, without amiguity of meaning. When S is finite, we can use a *multiplication table* to descibe the multiplication of S, similarly to finite groups.

Examples 1.2.2. • The set $\{0, 1\}$ together with the usual multiplication of real numbers forms a *finite* semigroup;

• The natural numbers $\mathbb{N}$ together with addition form a semigroup;

• The natural numbers $\mathbb{N}$ together with subtraction does *not* form a semigroup, as $1 - (2 - 3) = 2$, but $(1 - 2) - 3 = -4$. $\triangle$

As all groups have an associative binary operation, then every group is of course a semigroup. Every group also has an identity element, and hence is also a *monoid*, which we now define.

Definition 1.2.3. A *monoid* is a semigroup M containing an *identity* $1 \in M$, meaning for all $m \in M$,

$$1m = m1 = m. \qquad \blacksquare$$

It is a short exercise to see that identities in monoids are unique, hence we will always refer to *the* identity of M.

Examples 1.2.4. • The finite semigroup $(\{0,1\}, \times)$ is also a finite monoid, with 1 being the identity;

• $(\mathbb{N}, +)$ does *not* form a monoid, but $\mathbb{N} \cup \{0\}$ with addition *does*, taking 0 to be the identity;

• Every group is a monoid (and so every group is also a semigroup). $\triangle$

Of course, we can create Cartesian products of two semigroups or two monoids, and similarly define the direct product structure on them as follows.

Definition 1.2.5. For two semigroups (resp. monoids) S and T with operations $\cdot_S$ and $\cdot_T$ respectively, the *direct product of S and T* is the Cartesian product $S \times T$, together with the binary operation $\cdot : (S \times T) \times (S \times T) \to (S \times T)$ defined by

$$(s, t) \cdot (s', t') = (s \cdot_S s', t \cdot_T t'). \qquad \blacksquare$$

In exactly the same way as groups, we can generalise this to a direct product of a family of semigroups or monoids, as seen below.

Definition 1.2.6. The *direct product* of the family of semigroups (resp. monoids) $\{S_i\}_{i \in \mathcal{I}}$ is the Cartesian product $S = \prod_{i \in \mathcal{I}} S_i$ with operation $\cdot : S \times S \to S$, for which the product of two functions $f, g \in S$ is the function $f \cdot g : \mathcal{I} \to \bigcup_{i \in \mathcal{I}} S_i$ given by

$$(f \cdot g)(i) = f(i) \cdot_{S_i} g(i). \qquad \blacksquare$$

And, as expected, we have...

Proposition 1.2.7. *The direct product of two semigroups (resp. monoids) S and T is a semigroup (resp. monoid). The direct product of a family of semigroups (resp. monoids) $\{S_i\}_{i \in \mathcal{I}}$ is a semigroup (resp. monoid).* $\square$

1.3 Direct products in universal algebra

We can more broadly describe the structures we have seen so far as sets together with collections of operations satisfying certain properties. For example, a group has a *binary* operation (multiplication, taking two arguments), a *unary* operation (inversion, taking one argument) and a *nullary* operation (the identity, taking no arguments).

The study of sets with abstract collections of operations is known as *universal algebra*, whose chief objects of consideration are *algebras* (such as groups, semigroups, and monoids).

Definition 1.3.1. An *n-ary operation* on a set A is a map f from $\underbrace{A \times A \times \cdots \times A}_{n \text{ times}}$ to A. A *finitary operation* on A is an n-ary operation, for some $n \in \mathbb{N}$.

An *algebra* is a set A together with a collection F of finitary operations on A.

A *type of algebras* is a set $\mathcal{F}$ of map symbols $f \in \mathcal{F}$, for which a natural number n is assigned to each $f \in \mathcal{F}$. The number n is known as the *arity* of f. The *signature* of a type of algebras $\mathcal{F}$ is an ordered tuple of the arities of the map symbols of $\mathcal{F}$.

An algebra A is said to be of *type* $\mathcal{F}$ if, for every map symbol $f \in \mathcal{F}$ of arity n, there is a corresponding n-ary operation $f_A \in F$. ■

Examples 1.3.2.

- A group G comes equipped with the binary operation of multiplication, the unary operation of inversion, and the nullary operation which quantifies the existence of the identity.

  Giving these operations the map symbols $\cdot$, $^{-1}$, and 1 respectively, with arities 2, 1 and 0 then we can define the type $\mathcal{G} = \{\cdot, ^{-1}, 1\}$ with signature $(2, 1, 0)$. Groups are then algebras of type $\mathcal{G}$, but it is **not** necessarily the case that an algebra of type $\mathcal{G}$ is a group.

- A monoid has a binary operation and a nullary operation, so defining the set of map symbols $\mathcal{M} = \{\cdot, 1\}$ with signature $(2, 0)$, then monoids are algebras of type $\mathcal{M}$. Notice as well that the algebras of type $\mathcal{G}$ are also of type $\mathcal{M}$.

- Semigroups are algebras of type $\mathcal{S} = \{\cdot\}$ with signature (2). Algebras of type $\mathcal{G}$ or $\mathcal{M}$ are also of type $\mathcal{S}$. △

We can again frame direct products in this setting as follows.

Definition 1.3.3. If A and B are algebras of type $\mathcal{F}$ with collections of finitary operations F_A and F_B respectively, then the *direct product* of A and B is the set $A \times B$, with the collection of finitary operations $F_{A\times B}$ defined in the following way:

For every map symbol $f \in \mathcal{F}$ and corresponding n-ary operation $f_A \in F_A$ and $f_B \in F_B$, the n-ary operation $f_{A\times B} \in F_{A\times B}$ is defined by

$$f_{A\times B}((a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)) = (f_A(a_1, a_2, \ldots, a_n), f_B(b_1, b_2, \ldots, b_n)) \quad \blacksquare$$

This is quite a complicated generalisation of our previously "easy" direct product construction. Let us now see an example which compares the definitions.

Example 1.3.4. $\mathbb{N}$ with addition is a semigroup that is *not* a group, and $\mathbb{Z}$ with addition is a group. Even though $\mathbb{Z}$ is an algebra of type $\{\cdot, ^{-1}, 1\}$ (with signature $(2, 1, 0)$), $\mathbb{N}$ is not. But both are algebras of type $\{\cdot\}$ (with signature (2)).

Considering them as algebras of the same type, taking the finitary operations $\cdot_{\mathbb{N}}$ and $\cdot_{\mathbb{Z}}$ to both be addition, then we have our operations corresponding to the map symbol $\cdot$ of arity 2. Using the universal algebra definition of the direct product of $\mathbb{N}$ and $\mathbb{Z}$, letting $m, n \in \mathbb{N}$ and $p, q \in \mathbb{Z}$ be arbitrary, we get the binary operation $\cdot_{\mathbb{N}\times\mathbb{Z}}$ on $\mathbb{N} \times Z$ given by

$$\cdot_{\mathbb{N}\times\mathbb{Z}}((m, p), (n, q)) = (\cdot_{\mathbb{N}}(m, n), \cdot_{\mathbb{Z}}(p, q)) = (m + n, p + q).$$

In other words, we recover the expected operation on the direct product $\mathbb{N} \times \mathbb{Z}$, if we were to use the definition of a direct product of semigroups. $\triangle$

Chapter 2

Direct decomposability

When trying to determine the properties of a given algebraic structure, it is often useful to know if it can be *decomposed* as (or is isomorphic to) a direct product of related structures. The constituent factors of this direct product may give us some information about our original structure, or give a useful representation. For example, every finite Abelian group is isomorphic to a direct product of cyclic subgroups. This notion of *direct decomposition* is the topic of this chapter. We will consider what it means for a group or semigroup to be directly decomposable in particular.

2.1 Directly decomposing groups

Our motivation for this section is the following question: Given a group G, when is it isomorphic to a direct product of groups? Of course, we are more interested in knowing when this can be done in a non-trivial way, as $G \cong G \times \{1\}$. This motivates the following definition.

Definition 2.1.1. A group F is called *directly decomposable* if there exist non-trivial proper subgroups G, H of F such that $F \cong G \times H$. Otherwise, F is called *directly indecomposable.* ■

Examples 2.1.2.

- The Klein four-group K_4 is directly decomposable, by considering the subgroups $\{1, i\}$ and $\{1, j\}$ as isomorphic copies of $\mathbb{Z}_2$. That is, $K_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.
- $\mathbb{Z}_2$ is a simple group (having no non-trivial proper subgroups), and hence it is directly indecomposable.

- $\mathbb{Z}$ is *not* simple, but *is* directly indecomposable. As every non-trivial proper subgroup of $\mathbb{Z}$ is isomorphic to $\mathbb{Z}$ itself, it would have to be that $\mathbb{Z} \cong \mathbb{Z} \times \mathbb{Z}$ if $\mathbb{Z}$ were directly decomposable. This is of course a contradiction, as $\mathbb{Z} \times \mathbb{Z}$ is not cyclic. $\triangle$

Note that the direct product of groups G and H always contains the subgroups $G' = G \times \{1_H\}$ and $H' = \{1_G\} \times H$, which are isomorphic to G and H respectively. We will thus identify G with G', and H with H', and think of G and H as subgroups of $G \times H$. We note the following properties with this identification:

(1) The intersection $G \cap H$ contains only the identity of $G \times H$.

(2) Every element $(g, h) \in G \times H$ can be written as the product

$$(g, h) = (g, 1_H)(1_G, h).$$

This expression is a unique decomposition of (g, h) into a product of an element of G with an element of H.

(3) As $(g, 1_H)(1_G, h) = (1_G, h)(g, 1_H)$, then every element of G commutes with every element of H.

These three properties will play an important role when considering direct decomposability. Suppose that a group F has two subgroups G and H with these three properties *(1)-(3)* as given. Consider the mapping

$$\varphi : G \times H \to F,\ (g, h) \mapsto gh.$$

What can we say about it?

- φ is *well defined*, for if $(g, h) = (g', h')$, then $g = g'$, $h = h'$ and hence $\varphi(g, h) = gh = g'h' = \varphi(g', h')$.
- The map φ is *injective*, for if $\varphi(g, h) = \varphi(g', h')$, then $gh = g'h'$, and hence $(g')^{-1}g = h'h^{-1}$. But by property *(1)*, it must be that $(g')^{-1}g = h'h^{-1} = 1$, and hence $g = g', h = h'$ and $(g, h) = (g', h')$.
- The map φ is also *surjective* by property *(2)*, as any $f \in F$ can be written as $f = gh$ for some $g \in G$, $h \in H$, and hence $\varphi(g, h) = f$.
- Finally, φ is a *group homomorphism*, as for any $(g, h), (g', h') \in G \times H$, by using property *(3)* we have

$$\varphi((g, h)(g', h')) = \varphi(gg', hh') = gg'hh' = ghg'h' = \varphi(g, h)\varphi(g', h').$$

Thus φ is an isomorphism. We have now shown the following theorem.

Theorem 2.1.3. *Let F be a group, and let G and H be subgroups. If the following properties hold, then $F \cong G \times H$.*

(1) The intersection of G and H is trivial;

(2) Every element of F can be uniquely written as a product of an element of G with an element of H;

(3) Every element of G commutes with every element of H. $\square$

Conversely, given a direct product $F' = G' \times H'$ of groups G' and H', then there exist subgroups $G \cong G'$ and $H \cong H'$ of F' satisfying the above properties.

Note: Uniqueness isn't strictly required in property *(2)*, here. If $f \in F$ is such that $f = gh$ and $f = g'h'$ for some $g, g' \in G$, $h, h' \in H$, then $gh = g'h'$, implying $(g')^{-1}g = h'h^{-1}$. By property *(1)*, it must be that $(g')^{-1}g = h'h^{-1} = 1$, whence $g = g'$ and $h = h'$ giving uniqueness.

Example 2.1.4. Consider the dihedral group D_6, given by the presentation

$$D_6 = \langle \sigma, \rho : \sigma^6 = \rho^2 = (\sigma\rho)^2 = 1 \rangle.$$

Let's consider the subgroups $G = \{1, \sigma^3\}$ and $H = \{1, \rho, \sigma^2, \sigma^4, \sigma^2\rho, \sigma^4\rho\}$. Then $G \cap H = \{1\}$. The elements of $D_6 \setminus (G \cup H)$ are $\sigma, \sigma^5, \sigma\rho, \sigma^3\rho, \sigma^5\rho$. Using the identities in the presentation, we have

$$\sigma = \sigma^3\sigma^4, \;\; \sigma^5 = \sigma^3\sigma^2,$$
$$\sigma\rho = \sigma^3\sigma^4\rho, \;\; \sigma^5\rho = \sigma^3\sigma^2\rho.$$

and of course $\sigma^3\rho$ is the product of σ^3 and ρ, and hence every element of D_6 is a product of an element of G with an element of H.

Finally, noting that G is precisely the center of D_6, then every element of G commutes with every element of H. It thus follows by Theorem 2.1.3 that $D_6 \cong G \times H$. In particular, as $G \cong \mathbb{Z}_2$ and $H \cong D_3$, then $D_6 \cong \mathbb{Z}_2 \times D_3$. $\triangle$

A natural question is the following:

Question 2.1.5. Is every group isomorphic to a direct product of directly indecomposable groups?

We will answer this question when we come to study subdirect products.

2.2 Directly decomposing semigroups

In this section, we're using the same motivation as the previous section to ask the question: when is a semigroup isomorphic to a direct product of two semigroups? As for groups, we can define what it means for a semigroup to be directly decomposable, as in the following.

Definition 2.2.1. A semigroup S is called *directly decomposable* if there exist non-trivial semigroups T, U such that $S \cong T \times U$. Otherwise, S is called *directly indecomposable.* ■

The question of direct decomposability is more complicated than for the theory of groups, but it does generalise the result we obtained in the previous section. We first begin with some important background from semigroup theory to answer it.

2.2.1 Congruences on semigroups

We will use the theory of congruences on semigroups to characterise direct decomposability. We must first recall some of the theory of *equivalence relations* on sets.

Definition 2.2.2. A *binary relation* on a set X is a subset of $X \times X$. An *equivalence relation* $\sim$ on a set X is a subset $\sim \subseteq X \times X$ satisfying the following properties:

(R) Reflexivity: For all $x \in X$, $(x, x) \in \sim$;

(S) Symmetry: If $(x, y) \in \sim$, then $(y, x) \in \sim$;

(T) Transitivity: If $(x, y) \in \sim$ and $(y, z) \in \sim$, then $(x, z) \in \sim$.

The $\sim$-*equivalence class* of an element $x \in X$ is the set

$$[x]_\sim = \{y \in X : (x, y) \in \sim\}.$$

The *quotient set of S by $\sim$* is the set $X/\sim$ of all $\sim$-equivalence classes of X, given by

$$X/\sim = \{[x]_\sim \in \mathcal{P}(X) : x \in X\}.$$ ■

We will often write $x \sim y$ to mean $(x, y) \in \sim$, and say that x is "$\sim$-related" to y. This motivates the occasional use of notation such as "$=$" and "$\equiv$" as subsets.

Examples 2.2.3. • The subset $\Delta = \{(x, x) : x \in X\}$ of $X \times X$ is an equivalence relation on any set X, called the *equality relation on* X;

• The subset $\nabla = X \times X$ is an equivalence relation on any set X, called the *universal relation on* X;

• The subset $\equiv_2 = \{(x, y) \in \mathbb{N} \times \mathbb{N} : x \equiv y \pmod 2\}$ is an equivalence relation on $\mathbb{N}$;

• If X is a set of people, then the subset

$$\sim = \{(x, y) \in X \times X : x \text{ and } y \text{ have the same birthday}\}$$

is an equivalence relation on that set of people. △

Given two equivalence relations σ and ρ on the same set X, we can define their composition as follows.

Definition 2.2.4. Let σ, ρ be two equivalence relations on a set X. Then the *composition of* σ and ρ is the relation denoted $\sigma \circ \rho$ on X, defined by

$$(x, y) \in \sigma \circ \rho \Leftrightarrow \exists\, z \in X \text{ such that } (x, z) \in \sigma, (z, y) \in \rho.$$ ■

Note: The composition of two equivalence relations is not necessarily an equivalence relation.

Exercise 2.2.5. Let σ, ρ be two equivalence relations on a set X. Show that $\sigma \circ \rho$ is an equivalence relation on X if and only if $\sigma \circ \rho = \rho \circ \sigma$.

We can further use this composition to define the *join* of two equivalence relations, as follows.

Definition 2.2.6. Let σ, ρ be two equivalence relations on a set X. Then the *join* of σ and ρ is the relation denoted $\sigma \vee \rho$ on X, given by

$$(x, y) \in \sigma \vee \rho \Leftrightarrow \text{ For some } n \in \mathbb{N}, \exists\, x_1, x_2, \ldots, x_n \in X \text{ such that}$$
$$(x, x_1) \in \sigma, (x_1, x_2) \in \rho, (x_2, x_3) \in \sigma, \ldots, (x_{n-1}, x_n) \in \sigma, (x_n, y) \in \rho.$$

Equivalently

$$\sigma \vee \rho = (\sigma \circ \rho) \circ (\sigma \circ \rho) \circ (\sigma \circ \rho) \circ \ldots$$

Noting that $\sigma \circ \sigma = \sigma$ and $\rho \circ \rho = \rho$, then if $\sigma \circ \rho = \rho \circ \sigma$, it follows that $\sigma \vee \rho = \sigma \circ \rho$. ■

Lemma 2.2.7. *The join of two equivalence relations σ and ρ on a set X is an equivalence relation on X.*

Proof. Let $x, y, z \in X$ be arbitrary. Firstly, $(x, x) \in \sigma \vee \rho$, as $(x, x) \in \sigma$ and $(x, x) \in \rho$ by reflexivity, and hence taking $x_1 = x$ will do to show reflexivity of $\sigma \vee \rho$.

If $(x, y) \in \sigma \vee \rho$, then there exist $x_1, x_2, x_3, \dots, x_n \in X$ such that

$$(x, x_1) \in \sigma, (x_1, x_2) \in \rho, (x_2, x_3) \in \sigma, \dots, (x_{n-1}, x_n) \in \sigma, (x_n, y) \in \rho.$$

By symmetry and reflexivity of σ and ρ, then it is also true that

$$(y, y) \in \sigma, (y, x_n) \in \rho, (x_n, x_{n-1}) \in \sigma, \dots, (x_1, x) \in \sigma, (x, x) \in \rho.$$

So we see by reversing the sequence of x_i's that $(y, x) \in \sigma \vee \rho$ by definition, and $\sigma \vee \rho$ is symmetric.

Finally, if $(x, y), (y, z) \in \sigma \vee \rho$, then there exist $x_1, x_2, \dots, x_n, y_1, y_2, \dots y_m \in X$ such that

$$(x, x_1) \in \sigma, (x_1, x_2) \in \rho, (x_2, x_3) \in \sigma, \dots, (x_{n-1}, x_n) \in \sigma, (x_n, y) \in \rho, \tag{2.1}$$

and

$$(y, y_1) \in \sigma, (y_1, y_2) \in \rho, (y_2, y_3) \in \sigma, \dots, (y_{m-1}, y_m) \in \sigma, (y_m, z) \in \rho. \tag{2.2}$$

Reading (2.1) followed immediately by reading (2.2) gives that $(x, z) \in \sigma \vee \rho$ by definition, whence $\sigma \vee \rho$ is transitive, finishing the proof. □

We are now ready to define special types of equivalence relations on semigroups called *congruences*, which will play an important role in direct decomposability for semigroups.

Definition 2.2.8. For a semigroup S, a *congruence* on S is an equivalence relation σ on S such that

$$(s, t) \in \sigma \text{ and } (u, v) \in \sigma \Rightarrow (su, tv) \in \sigma.$$

In this way, σ is said to be *compatible* with the multiplication of S. ■

Examples 2.2.9. • The equality relation $\Delta = \{(s,s) \in S \times S\}$ from Examples 2.2.3 is an equivalence relation on any semigroup S, and moreover if $(s,s) \in \Delta$, and $(t,t) \in \Delta$, then $(st, st) \in \Delta$. Hence the equality relation is a congruence on any semigroup S;

• Similarly, the universal relation $\nabla = S \times S$ is a congruence on any semigroup S, as $(s,t) \in \nabla$ and $(u,v) \in \nabla$ imply that $s, t, u, v \in S$, and hence $su, tv \in S$, so that $(su, tv) \in \nabla$;

• Given a semigroup homomorphism $\varphi : S \to T$, consider its *kernel*, defined

$$\ker \varphi = \{(s,t) \in S \times S : \varphi(s) = \varphi(t)\}.$$

It is an equivalence relation on S: being clearly reflexive, symmetric and transitive by equality of images. Further, if $(s,t), (u,v) \in \ker \varphi$, then $\varphi(s) = \varphi(t)$ and $\varphi(u) = \varphi(v)$, hence

$$\varphi(su) = \varphi(s)\varphi(u) = \varphi(t)\varphi(v) = \varphi(tv),$$

and $(su, tv) \in \ker \varphi$. Hence kernels of homomorphisms are congruences. $\triangle$

Congruences on groups will probably be familiar to you already, even if you don't recognise them just yet. That's because, in fact, there is a 1-1 correspondence between group congruences and normal subgroups. We prove this now with the following two lemmas.

Lemma 2.2.10. *Let G be a group, and let $\sim$ be a congruence on G. Then $N = [1]_\sim$ is a normal subgroup of G, and $(g,h) \in \sim$ if and only if $g^{-1}h \in N$.*

Proof. We will first show that $[1]_\sim$ is a subgroup, using the subgroup test. Firstly, N is a non-empty subset of G, as $1 \in N$. If $g, h \in N$ then $(1,g), (1,h) \in \sim$. As $\sim$ is reflexive, then $(h^{-1}, h^{-1}) \in \sim$, and further as $\sim$ is a congruence, then in particular $(1h^{-1}, hh^{-1}) = (h^{-1}, 1) \in \sim$. By symmetry, $(1, h^{-1}) \in \sim$, so

$$(1^2, gh^{-1}) = (1, gh^{-1}) \in \sim,$$

and hence $gh^{-1} \in N$. Hence N is a subgroup of G. To show that N is normal in G, let $g \in G, n \in N$. Then $(g,g), (g^{-1}, g^{-1}) \in \sim$ by reflexivity, and $(1,n) \in \sim$ by definition. Hence

$$(g^{-1}1g, g^{-1}ng) = (1, g^{-1}ng) \in \sim,$$

so that $g^{-1}ng \in N$. Hence N is a normal subgroup. Finally, using the property of congruences,

$$(g,h) \in \sim \Leftrightarrow (g^{-1}g, g^{-1}h) \in \sim \Leftrightarrow (1, g^{-1}h) \in \sim \Leftrightarrow g^{-1}h \in N. \qquad \square$$

Lemma 2.2.11. *Let G be a group, and let N be a normal subgroup of G. Then N induces a congruence $\sim$ on G, defined*

$$(g,h) \in \sim \Leftrightarrow g^{-1}h \in N.$$

Proof. We will show first that $\sim$ is an equivalence relation. Firstly, as N is a normal subgroup of G, it contains the identity, and hence $1 = g^{-1}g \in N$ for any $g \in G$. Thus $(g,g) \in \sim$ for any $g \in G$, and hence $\sim$ is reflexive.

Secondly, if $(g,h) \in \sim$, then $g^{-1}h \in N$ by definition. As N is a subgroup, then the inverse $(g^{-1}h)^{-1} = h^{-1}g \in N$, and hence $(h,g) \in \sim$, so $\sim$ is symmetric.

Thirdly, if $(g,h),(h,i) \in \sim$, then $g^{-1}h, h^{-1}i \in N$, and so the product $g^{-1}hh^{-1}i = g^{-1}i \in N$, so that $(g,i) \in \sim$. Thus $\sim$ is transitive, and hence an equivalence relation on G.

Finally, to show that $\sim$ is a congruence on G, let $(g,h),(i,j) \in \sim$. Then $g^{-1}h, i^{-1}j \in N$. In particular as N is normal in G, then $j^{-1}g^{-1}hj \in N$. As N is a subgroup of G, then we have

$$(i^{-1}j)(j^{-1}g^{-1}hj) = i^{-1}g^{-1}hj = (gi)^{-1}hj \in N,$$

and hence $(gi, hj) \in \sim$ as required. $\square$

Returning now to congruences in semigroup theory, let's see some operations on congruences in the following lemma.

Lemma 2.2.12. *Given two congruences σ, ρ on a semigroup S, the join $\sigma \vee \rho$ and intersection $\sigma \cap \rho$ are also congruences on S.*

Proof. That $\sigma \vee \rho$ is an equivalence relation has already been shown. If $(s,t),(u,v) \in \sigma \vee \rho$, then there exist elements $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m \in S$ such that

$$(s,x_1) \in \sigma, (x_1,x_2) \in \rho, (x_2,x_3) \in \sigma, \ldots, (x_{n-1},x_n) \in \sigma, (x_n,t) \in \rho,$$

and

$$(u, y_1) \in \sigma, (y_1, y_2) \in \rho, (y_2, y_3) \in \sigma, \ldots, (y_{m-1}, y_m) \in \sigma, (y_m, v) \in \rho.$$

We can also assume that $m = n$ without losing any generality, for if $m < n$, then we can let $y_m = y_{m+1} = y_{m+2} = \cdots = y_n$, and hence by reflexivity of σ and ρ we can still obtain elements $y_1, y_2, \ldots y_n \in S$ such that

$$(u, y_1) \in \sigma, (y_1, y_2) \in \rho, \ldots, (y_{m-1}, y_m) \in \rho,$$

$$(y_m, y_{m+1}) \in \sigma, (y_{m+1}, y_{m+2}) \in \rho \ldots, (y_{n-1}, y_n) \in \sigma, (y_n, v) \in \rho$$

and similarly if $m > n$. As σ and ρ are congruences, it follows that

$$(su, x_1y_1) \in \sigma, (x_1y_2, x_2y_2) \in \rho, \ldots, (x_{n-1}y_{n-1}, x_ny_n) \in \sigma, (x_ny_n, tv) \in \rho,$$

and hence that $(su, tv) \in \sigma \vee \rho$. Thus $\sigma \vee \rho$ is a congruence.

In the case of $\sigma \cap \rho$, as $(x, y) \in \sigma \cap \rho \Leftrightarrow (x, y) \in \sigma, (x, y) \in \rho$, then it quickly follows that $\sigma \cap \rho$ is a congruence from σ and ρ being congruences. □

Given a congruence σ on a semigroup S, we are able to give the quotient set S/σ an algebraic structure, similarly to how quotients of groups by normal subgroups are obtained.

Lemma 2.2.13. *Let S be a semigroup, and let σ be a congruence on S. Then the quotient set*

$$S/\sigma = \{[s]_\sigma : s \in S\},$$

together with the operation $\cdot : S/\sigma \to S/\sigma$ defined

$$[s]_\sigma \cdot [t]_\sigma = [st]_\sigma$$

is a semigroup. □

Definition 2.2.14. For a semigroup S and a congruence σ on S, the quotient set S/σ together with the operation $\cdot : S/\sigma \to S/\sigma$ defined

$$[s]_\sigma \cdot [t]_\sigma = [st]_\sigma$$

is called the *quotient semigroup of S by σ.* ■

Finally, we now give a generalisation of the first isomorphism theorem of groups, fpr which we leave the proof as an exercise to the reader.

Theorem 2.2.15 (First Isomorphism Theorem (for semigroups))**.** *Let S, T be semigroups, and let $\varphi : S \to T$ be a homomorphism. Then* $\ker\varphi$ *is a congruence on S,* $\operatorname{im}\varphi$ *is a subsemigroup of T, and*

$$S/\ker\varphi \cong \operatorname{im}\varphi$$

Proof. Exercise. □

2.2.2 A characterisation of direct decomposition for semigroups

We now have the tools necessary to consider directly decomposing semigroups. Let's first consider the case when we have a direct product $P = S \times T$ of semigroups S and T given to us.

There are two natural semigroup homomorphisms when given such a direct product: the projection maps from P to S and from P to T. Formally, these are the maps

$$\pi_S : P \to S,\ (s,t) \mapsto s$$
$$\pi_T : P \to T,\ (s,t) \mapsto t.$$

As we saw in Examples 2.2.9, the kernels of these homomorphisms are congruences on P. Noting that

$$((s,t),(u,v)) \in \ker\pi_S \Leftrightarrow \pi_S(s,t) = \pi_S(u,v) \Leftrightarrow s = u,$$

and similarly that

$$((s,t),(u,v)) \in \ker\pi_T \Leftrightarrow \pi_S(s,t) = \pi_S(u,v) \Leftrightarrow t = v,$$

then it follows that $((s,t),(u,v)) \in \ker\pi_S \cap \ker\pi_T \Leftrightarrow (s,t) = (u,v)$, and hence the intersection $\ker\pi_S \cap \ker\pi_T$ is equal to the equality relation Δ that we saw in Examples 2.2.9.

What about the composition of these congruences? Well

$$\begin{aligned}((s,t),(u,v)) \in \ker\pi_S \circ \ker\pi_T \Leftrightarrow\ & \exists (x,y) \in P \text{ such that } ((s,t),(x,y)) \in \ker\pi_S \\ & \text{and } ((x,y),(u,v)) \in \ker\pi_T \\ \Leftrightarrow\ & (s,v) \in P\end{aligned}$$

But as P is the direct product of S and T, then this last condition is certainly true, and hence it follows that $((s,t),(u,v)) \in \ker \pi_S \circ \ker \pi_T$ for all $(s,t),(u,v) \in P$. Thus $\ker \pi_S \circ \ker \pi_T$ is equal to the universal relation ∇ that we saw in Examples 2.2.9. Similarly, we can show that

$$((s,t),(u,v)) \in \ker \pi_T \circ \ker \pi_S \Leftrightarrow (u,t) \in P,$$

and hence that $\ker \pi_T \circ \ker \pi_S$ is also equal to the universal relation ∇. In particular, notice that

$$\ker \pi_S \circ \ker \pi_T = \ker \pi_T \circ \ker \pi_S.$$

Finally, what about the join of these congruences? Well any pair $((s,t),(u,v))$ from $P \times P$ is indeed in the join $\ker \pi_S \vee \ker \pi_T$, as

$$((s,t),(s,v)) \in \ker \pi_S, ((s,v),(u,v)) \in \ker \pi_T,$$

and so it must again be that the join $\ker \pi_S \vee \ker \pi_T$ is equal to the universal relation ∇.

We have proved the following lemma.

Lemma 2.2.16. *Let $P = S \times T$ be the direct product of semigroups S and T. Then there exist congruences σ, ρ on P such that*

(1) $\sigma \cap \rho = \Delta$;

(2) $\sigma \circ \rho = \rho \circ \sigma$;

(3) $\sigma \vee \rho = \nabla$. □

Now conversely, given any semigroup S and two congruences σ and ρ on S matching the conditions of Lemma 2.2.16, we claim that S is isomorphic to a direct product. We prove this result in the next lemma.

Lemma 2.2.17. *Let S be a semigroup, and let σ, ρ be congruences on S such that $\sigma \cap \rho = \Delta$, $\sigma \circ \rho = \rho \circ \sigma$ and $\sigma \vee \rho = \nabla$. Then*

$$S \cong (S/\sigma) \times (S/\rho).$$

Proof. We define the mapping $\varphi : S \to (S/\sigma) \times (S/\rho)$ by $\varphi(s) = ([s]_\sigma, [s]_\rho)$. Then φ is well defined, for if $s = t$, then $[s]_\sigma = [t]_\sigma$ and similarly $[s]_\rho = [t]_\rho$, and hence $\varphi(s) = ([s]_\sigma, [s]_\rho) = ([t]_\sigma, [t]_\rho) = \varphi(t)$.

Conversely, if $\varphi(s) = \varphi(t)$ for some $s, t \in S$, then $([s]_\sigma, [s]_\rho) = ([t]_\sigma, [t]_\rho)$, whence

$[s]_\sigma = [t]_\sigma$ and $[s]_\rho = [t]_\rho$. It follows then that $(s,t) \in \sigma \cap \rho$, which must mean $s = t$ as $\sigma \cap \rho$ is equal to the equality relation. Thus φ is injective.

To show that φ is a surjection, then let $([s]_\sigma, [t]_\rho) \in (S/\sigma) \times (S/\rho)$ be arbitrary. As $\sigma \vee \rho = \nabla$, then in particular $(s,t) \in \sigma \vee \rho$. As $\sigma \circ \rho = \rho \circ \sigma$, then recall that implies $\sigma \vee \rho = \sigma \circ \rho$. Hence there exists some $u \in S$ such that $(s,u) \in \sigma$ and $(u,t) \in \rho$. That is, $[s]_\sigma = [u]_\sigma$, $[t]_\rho = [u]_\rho$. Thus

$$([s]_\sigma, [t]_\rho) = ([u]_\sigma, [u]_\rho) = \varphi(u),$$

and hence φ is a surjection

Finally, for any $s, t \in S$, we have

$$\varphi(st) = ([st]_\sigma, [st]_\rho) = ([s]_\sigma [t]_\sigma, [s]_\rho [t]_\rho) = ([s]_\sigma, [s]_\rho)([t]_\sigma, [t]_\rho) = \varphi(s)\varphi(t)$$

Hence φ is an isomorphism, and the result follows. □

Note: Throughout this section, we could have used groups, monoids, or algebras instead of semigroups (associativity was not necessarily used). The above result is true more generally in universal algebra.

Chapter 3

Introduction to subdirect products

When we discussed direct decomposability, we posed the question "Is every group isomorphic to a direct product of directly indecomposable groups?". More generally, we will ask "Is every semigroup isomorphic to a direct product of directly indecomposable semigroups?". If the semigroup is finite, the answer is yes, as seen in the following.

Proposition 3.0.1. *Every finite semigroup S is isomorphic to a direct product of directly indecomposable semigroups.*

Proof. We will proceed by induction on $|S|$. For the base case when $|S| = 1$, S itself is trivial, and hence directly indecomposable, so there is nothing to show. Hence suppose S is a non-trivial semigroup (that is, $|S| > 1$). Assume for the inductive hypothesis that, for all finite semigroups T with $|T| < |S|$, T is isomorphic to a direct product of directly indecomposable semigroups.

If S is directly indecomposable, there is nothing to show. Otherwise, S is directly decomposable, with $S \cong T \times U$ for some non-trivial semigroups T and U. As $|S| = |T \times U| = |T||U|$, and T and U are non-trivial, then it must be that $|T| < |S|$ and $|U| < |S|$. By the inductive hypothesis, it follows that T is isomorphic to a direct product of directly indecomposable semigroups, as is U. Thus S is, by taking the direct product of the decompositions for T and U. Hence we've proved our inductive hypothesis, and thus the result. □

The result is not true in general, however (a counter-example being any countably infinite Boolean algebra). We thus seek a modification of the direct product notion for which a statement akin to' "Every semigroup is isomorphic to a *something* product of *something*-ly indecomposable semigroups" is true. The correct notion is that of subdirect products, for which we now define some preliminaries.

3.1 Preliminaries

Definition 3.1.1. Let S and T be two semigroups. A *subdirect product* of S with T (sometimes referred to as a subdirect product of $S \times T$) is a subsemigroup U of the direct product $S \times T$, such that the projection maps defined

$$\pi_S : U \to S := (s,t) \mapsto s,$$
$$\pi_T : U \to T := (s,t) \mapsto t$$

are surjections. In this case, we will write $U \leq_{\mathrm{sd}} S \times T$. ■

Again, we can define a subdirect product of a family of semigroups $\{S_i\}_{i\in\mathcal{I}}$ to be a subsemigroup S of the direct product $\prod_{i\in\mathcal{I}} S_i$ of that family, where each projection mapping $\pi_i : S \to S_i$ onto the i-th factor is a surjection. Further, there was no particular reason to pick semigroup here: subdirect products of groups, monoids, and other algebras are similarly defined.

Examples 3.1.2. • The direct product $S \times T$ itself is a subdirect product of semigroups S and T, as every element of S appears in some first coordinate, and every element of T appears in some second coordinate. Hence the projection maps onto S and T are surjections;

• The semigroup given by $\Delta_S := \{(s,s) : s \in S\}$ is a subdirect product of a semigroup S with itself, as every element of S appears in some first and second coordinate. Δ_S is known as the *diagonal* subdirect product;

• Let $L = \{0,1\}$ be the set of real numbers under the usual multiplication, and let S be the subsemigroup of $L \times L$ given by $S = \{(1,0),(0,1),(0,0)\}$. This is a subdirect product, as both $0,1$ appear in the first coordinate of some pair, and similarly for the second coordinates, and hence the projection maps are surjective. △

We will think of being a subdirect product as more of a property of a semigroup rather than a direct construction. A particular type of constructible subdirect product is a *fiber product* of semigroups, which we now define.

Definition 3.1.3. Given semigroups S, T, U and epimorphisms $\varphi : S \to U$,

$\psi : T \to U$, the *fiber product of S and T with respect to φ, ψ* is the set

$$\Pi(\varphi, \psi) := \{(s,t) \in S \times T : \varphi(s) = \psi(t)\}$$

with multiplication inherited from $S \times T$. U is called the *fiber*, or *fiber quotient* of $\Pi(\varphi, \psi)$. If V is a subdirect product of $S \times T$ which is also a fiber product, we will write $V \leq_{\text{fp}} S \times T$. ■

Again, there was no particular reason to define this only for semigroups, and hence fiber products of other algebras are similarly defined. We now qualify in the following lemma that fiber products are indeed examples of subdirect products.

Lemma 3.1.4. *For semigroups S, T, U and epimorphisms $\varphi : S \to U$, $\psi : T \to U$, $\Pi(\varphi, \psi)$ is a subdirect product of $S \times T$.*

Proof. If (s,t), $(s',t') \in \Pi(\varphi, \psi)$, then $(s,t)(s',t') = (ss', tt') \in \Pi(\varphi, \psi)$, as

$$\varphi(ss') = \varphi(s)\varphi(s') = \psi(t)\psi(t') = \psi(tt'),$$

and hence $\Pi(\varphi, \psi)$ is a subsemigroup of $S \times T$. Moreover, for any $s \in S$, as $\varphi(s) \in U$ and ψ is surjective, there exists some $t \in T$ such that $\varphi(s) = \psi(t)$, and hence $(s,t) \in \Pi(\varphi, \psi)$. Similarly, for any $t \in T$, as φ is surjective, there exists some $s \in S$ such that $\psi(t) = \varphi(s)$, and hence $(s,t) \in \Pi(\varphi, \psi)$. Thus the projection maps

$$\pi_S : \Pi(\varphi, \psi) \to S := (s,t) \mapsto s,$$
$$\pi_T : \Pi(\varphi, \psi) \to T := (s,t) \mapsto t.$$

are surjections, and $\Pi(\varphi, \psi)$ is a subdirect product of $S \times T$. □

Examples 3.1.5. • For semigroup S and T, the direct product $S \times T$ is a fiber product. We take U to be the trivial group $\{1\}$, φ to be the constant map given by

$$\varphi : S \to U := \varphi(s) = 1,$$

and similarly ψ to be the constant map given by

$$\psi : T \to U := \psi(t) = 1.$$

• For a semigroup S, the diagonal subdirect product $\Delta_S = \{(s,s) : s \in S\}$ is a fiber product. We take $U = S$, $\varphi : S \to S$ to be the identity mapping (so that

$\varphi(s) = s$), and $\psi = \varphi$. Then $\varphi(s) = \psi(t) \Leftrightarrow s = t$. $\triangle$

It is a natural question to ask which subdirect products can be constructed in this way. The answer is given by Fleischer's lemma, as follows.

Lemma 3.1.6 (Fleischer's lemma, [4, Lemma 10.1])**.** *Let S, T, U be semigroups, and let $U \leq_{\mathrm{sd}} S \times T$. For the projection maps*

$$\pi_S : U \to S := (s,t) \mapsto s,$$
$$\pi_T : U \to T := (s,t) \mapsto t,$$

denote by σ the congruence $\ker \pi_S$ on U, and denote by ρ the congruence $\ker \pi_T$ on U. Then U is a fiber product of S with T if and only if

$$\sigma \circ \rho = \rho \circ \sigma.$$

Proof. ($\Rightarrow$) If U is a fiber product of S with T, then there exist epimorphisms $\varphi : S \to V$, $\psi : T \to V$ onto a common image V, such that

$$U = \{(s,t) \in S \times T : \varphi(s) = \psi(t)\}.$$

Let $((s_1,t_1),(s_2,t_2)) \in \sigma \circ \rho$. Then there exists $(s_3,t_3) \in U$ such that

$$((s_1,t_1),(s_3,t_3)) \in \sigma, \quad ((s_3,t_3),(s_2,t_2)) \in \rho.$$

By the definitions of σ and ρ, it must be that $s_1 = s_3$, and $t_3 = t_2$. Hence $(s_1,t_2) \in U$, and thus $\varphi(s_1) = \psi(t_2)$. As $(s_1,t_1) \in U$ and $(s_2,t_2) \in U$, it follows that

$$\varphi(s_2) = \psi(t_2) = \varphi(s_1) = \psi(t_1),$$

and hence $(s_2,t_1) \in U$ also. As

$$((s_1,t_1),(s_2,t_1)) \in \rho, \quad ((s_2,t_1),(s_2,t_2)) \in \sigma,$$

then $((s_1,t_1),(s_2,t_2)) \in \rho \circ \sigma$. As $((s_1,t_1),(s_2,t_2))$ was an arbitrary element of $\sigma \circ \rho$, then we have shown that $\sigma \circ \rho \subseteq \rho \circ \sigma$. The reverse inclusion follows by a symmetric argument, and hence

$$\sigma \circ \rho = \rho \circ \sigma.$$

($\Leftarrow$) If $\sigma \circ \rho = \rho \circ \sigma$, then $\sigma \circ \rho = \sigma \vee \rho$ is a congruence. Let $\iota : U \to U/(\sigma \circ \rho)$

be the natural quotient mapping $\iota(u,v) = [(u,v)]_{\sigma\circ\rho}$. As $\sigma \subseteq \sigma\circ\rho$, there is a natural epimorphism from U/σ to $U/(\sigma\circ\rho)$ given by

$$\pi : U/\sigma \to U/(\sigma\circ\rho) := [(u,v)]_\sigma \mapsto [(u,v)]_{\sigma\circ\rho}.$$

As U is a subdirect product, then

$$S \cong U/\sigma$$

(by the first isomorphism theorem). Hence there exists an epimorphism $\varphi : S \to U/(\sigma\circ\rho)$ with $\iota = \pi_S \circ \varphi$ (recalling that our convention is to compose from left to right). A similar proof shows that there also exists an epimorphism $\psi : T \to U/(\sigma\circ\rho)$ with $\iota = \pi_T \circ \psi$. We have the following commuting diagram.

$$\begin{tikzcd}
U \arrow[r, two heads, "\pi_T"] \arrow[d, two heads, "\pi_S"] \arrow[dr, dashed, "\iota"] & T \arrow[d, two heads, "\psi"] \\
S \arrow[r, two heads, "\varphi"] & U/(\sigma\circ\rho)
\end{tikzcd}$$

We claim that

$$U = \{(u,v) \in S \times T : \varphi(u) = \psi(v)\}$$

If $(u,v) \in U$, then $\iota(u,v) = (\pi_S \circ \varphi)(u,v) = \varphi(u)$, and $\iota(u,v) = (\pi_T \circ \psi)(u,v) = \psi(v)$. Hence $\varphi(u) = \psi(v)$.

Conversely, if $(u,v) \in S \times T$ is such that $\varphi(u) = \psi(v)$, then as U is a subdirect product of S with T, there exist $s \in S$ and $t \in T$ such that $(u,t) \in U$ and $(s,v) \in U$. Now

$$\iota(u,t) = (\pi_S \circ \varphi)(u,v) = \varphi(u) = \psi(v) = (\pi_T \circ \psi)(s,v) = \iota(s,v),$$

and hence $((u,t),(s,v)) \in \sigma\circ\rho$. Hence there must exist $(x,y) \in U$ such that $((u,t),(x,y)) \in \sigma$ and $((x,y),(s,v)) \in \rho$. This implies however that $x = u$ and $y = v$, and hence $(u,v) \in U$. Thus we have shown that

$$U = \{(u,v) \in S \times T : \varphi(u) = \psi(v)\},$$

and U is a fiber product. $\square$

For semigroups, there are hence examples of subdirect products which are not fiber products, as seen below.

Example 3.1.7. We saw that for $L = \{0, 1\}$, the subsemigroup

$$S = \{(0, 1), (1, 0), (0, 0)\}$$

of $L \times L$ was a subdirect product. Using the notation of Fleischer's lemma, recall for pairs $(s, t), (u, v) \in S$ that

$$((s, t), (u, v)) \in \sigma \circ \rho \Leftrightarrow (s, v) \in S,$$

and similarly that

$$((s, t), (u, v)) \in \rho \circ \sigma \Leftrightarrow (u, t) \in S.$$

Now $((0, 1), (1, 0)) \in \sigma \circ \rho$ as $(0, 0) \in S$, however $((0, 1), (1, 0)) \notin \rho \circ \sigma$, as $(1, 1) \notin S$. Hence $\sigma \circ \rho \neq \rho \circ \sigma$, and thus S cannot be a fiber product by Fleischer's lemma. △

3.2 Subdirect products of groups

For a subdirect product K of groups G and H, for any $(g, h), (g', h') \in K$, recall that $((g, h), (g', h')) \in \sigma \circ \rho$ if and only if $(g, h') \in K$. As K is a subgroup of $G \times H$, then this implies $(g, h')^{-1} = (g^{-1}, (h')^{-1}) \in K$, and thus

$$(g, h)(g^{-1}, (h')^{-1})(g', h') = (g', h) \in K.$$

Recalling that $((g, h), (g', h')) \in \rho \circ \sigma$ if and only if $(g', h) \in K$, then we have shown that

$$((g, h), (g', h')) \in \sigma \circ \rho \Rightarrow ((g, h), (g', h')) \in \rho \circ \sigma,$$

and hence that $\sigma \circ \rho \subseteq \rho \circ \sigma$. A similar consideration will show that $\rho \circ \sigma \subseteq \sigma \circ \rho$, and hence that $\sigma \circ \rho = \rho \circ \sigma$. We have thus shown the following

Proposition 3.2.1. *Every subdirect product of groups can be constructed as a fiber product.* □

A more concrete description of the subdirect products of two groups is given by Goursat's lemma, which we present without proof.

Corollary 3.2.2 (Goursat's lemma)**.** *Let G and G' be groups, let $N \trianglelefteq G$ and $N' \trianglelefteq G'$ be normal subgroups. Given an isomorphism $\varphi : G/N \to G'/N'$, then*

φ induces a subdirect product

$$H_{\varphi} = \{(g, g') \in G \times G' : \varphi(gN) = g'N'\}$$

of G and G'. Moreover, every subdirect product H of G and G' is obtained in this way. □

3.3 Subdirect reducibility & Birkhoff's representation theorem

Having now met with subdirect products, we can define what it means for a semigroup (or indeed, group, monoid, or other algebra) to be *subdirectly irreducible*. The concept is similar to that of direct indecomposability, although this time we will use the infinite definition of direct product as seen in the following definition.

Definition 3.3.1. A *subdirect representation* of a semigroup S is subdirect product T of a family of semigroups $\{S_i\}_{i\in\mathcal{I}}$ for which T is isomorphic to S.

A semigroup S is said to be *subdirectly irreducible* if, for any subdirect representation $T \leq_{\text{s.d}} \prod_{i\in\mathcal{I}} S_i$, one of the S_i is isomorphic to S. Otherwise, S is said to be *subdirectly reducible* ■

Again, this definition could have been given for algebras more generally. We now give the answer to the question "is every semigroup isomorphic to a subdirect product of subdirectly irreducible semigroups" in the positive, by stating Birkhoff's theorem (without proof) for algebras.

Theorem 3.3.2 (Birkhoff's representation theorem , [4, Theorem 8.6])**.** *Every algebra A is isomorphic to a subdirect product of subdirectly irreducible algebras.*

□

This result is our main motivation for considerations of subdirect products. They can provide a potentially useful tool for representing algebras; particularly when they can be constructed as fiber products.

Chapter 4

Finitary properties for direct and subdirect products

We motivated the direct product and subdirect product definitions with a desire to be able to consider properties of a given algebra using properties of the factors of a representation constituting it. Conversely, we'd often like to be able to construct examples of algebras with a set of given properties, using direct and subdirect products of families of algebras whose properties we know. Properties that we will pay particular interest to in this chapter will be *finitary* properties.

Although we will see that the direct product of groups is generally well behaved when it comes to inheriting finitary properties from its factors, we will see a number of interesting examples of subgroups of direct products which do not inherit such properties. Subdirect products of groups in particular have often been used as a source of interesting and unexpected counterexamples in this regard. We finish these lecture notes with an overview of some of these classic examples and results in both group theory and semigroup theory, as well as those stemming from recent research. Further, we will talk about some of the more general open research questions surrounding the theory of subdirect products in the context of universal algebra in particular.

Being a descriptive chapter, there will be a number of concepts from the theory of finitary properties which we have not seen in these lectures, which will not be covered in this chapter. We highly recommend the interested reader to divulge in some of the recommended reading (and further for the avid researcher; the papers themselves) that follows these lecture notes.

4.1 Finitary properties in direct products

Given an infinite algebraic structure, we would ideally like to be able to represent it in some finite manner. For example, can we build every element of that structure using only a finite subset of elements? Can we describe any relations governing the structure from a finite number of relations? These are the ideas of *finite generation* and *finite presentation* that play an important role in computational algebra. Finite presentations for groups, for example, provide us with a finite set of symbols and relations on which symbolic computation can be performed.

These properties are examples of *finitary properties*, which we now define.

Definition 4.1.1. For a class of algebras, a *finitary property* is one which is satisfied for all finite algebras in that class. ■

Examples 4.1.2.

- Being a finite algebra is of course a finitary property;
- The ascending chain condition (and descending chain condition) on algebras with a partial ordering is a finitary property, as any finite algebra has a finite power set;
- Residual finiteness for groups (meaning the group can be embedded in a direct product of a family finite groups) is a finiteness condition, as every finite group can be embedded in itself;
- Being finitely generated for groups & semigroups (meaning every element can be expressed as a finite product of elements from a finite subset of the structure) is a finitary property, as the entire finite group or semigroup can be used as a generating set;
- Local finiteness for groups (meaning every finitely generated subgroup is finite) is a finiteness property, as indeed, every subgroup of a finite group is necessarily finite;
- Similarly, finite presentability for groups & semigroups is a finiteness condition, taking the relations of the presentation for the finite group or semigroup to be the entire multiplication table. △

Given an algebra with a finitary property P and a direct product representation for that algebra, do the factors of that representation also have property P? Conversely, give two algebras of the same type, each with finitary property P, does the direct product also have property P?

The latter is the concept of closure under direct products, and has been a question of much interest classically. We will look at the groups case to begin.

4.1.1 Finitary properties for direct products of groups

For groups, we seek finitary properties P for which the statement

Groups G and H have property P if and only if $G \times H$ has property P

holds, and indeed there are many. We will highlight the proofs for three finitary properties in particular.

Proposition 4.1.3. *Groups G and H are residually finite if and only if $G \times H$ is residually finite.*

Proof. ($\Rightarrow$) If G is residually finite, there exists a family $\{G_i\}_{i \in \mathcal{I}}$ of finite groups, and an embedding $\varphi : G \to \prod_{i \in \mathcal{I}} G_i$. Similarly, there exists a family $\{H_j\}_{j \in \mathcal{J}}$ of finite groups, and an embedding $\psi : H \to \prod_{j \in \mathcal{J}} H_j$. The mapping θ from $G \times H$ to the direct product $\prod_{i \in \mathcal{I}} G_i \times \prod_{j \in \mathcal{J}} H_j$ given by $\theta(g, h) = (\varphi(g), \psi(h))$ is verifiably an embedding,

Moreover, without loss of generality, we can assume that $\mathcal{I}$ and $\mathcal{J}$ are disjoint. So taking the disjoint union $\mathcal{L} = \mathcal{I} \sqcup \mathcal{J}$ of these indexing sets, we obtain the new family of finite groups $\{K_l\}_{l \in \mathcal{L}}$, where

$$K_l = \begin{cases} G_l & \text{if } l \in \mathcal{I}, \\ H_l & \text{if } l \in \mathcal{J}. \end{cases}$$

Note that the direct product $\prod_{i \in \mathcal{I}} G_i \times \prod_{j \in \mathcal{J}} H_j$ is embeddable in $\prod_{l \in \mathcal{L}} K_l$, by taking the map $\iota : \prod_{i \in \mathcal{I}} G_i \times \prod_{j \in \mathcal{J}} H_j \to \prod_{l \in \mathcal{L}} K_l$ given by $\iota(f, g) = h$ where $h : \mathcal{L} \to \bigcup_{l \in \mathcal{L}} K_l$ is defined by

$$h(l) = \begin{cases} f(l) & \text{if } l \in \mathcal{I} \\ g(l), & \text{if } l \in \mathcal{J}. \end{cases}$$

Now the composition $\theta \circ \iota$ is also an embedding, and hence we have embedded the direct product $G \times H$ into a family $\{K_l\}_{l \in \mathcal{L}}$ of finite groups. Hence the direct product is residually finite.

($\Leftarrow$) As G is isomorphic to $G \times \{1_H\}$, it is certainly embeddable in $G \times \{1_H\}$.

$G \times \{1_H\}$ can clearly be embedded in $G \times H$, which in turn can be embedded in a direct product of a family of finite groups. The composition of these embeddings will give an embedding of G into a direct product of a family of finite groups, and hence G is residually finite. A similar argument also works for H. $\square$

Proposition 4.1.4. *Groups G and H are locally finite if and only if $G \times H$ is locally finite.*

Proof. ($\Rightarrow$) If K is a finitely generated subgroup of $G \times H$, then the image $\pi_G(K)$ is a finitely generated subgroup of G, where

$$\pi_G : G \times H \to G := (g, h) \mapsto g.$$

Hence $\pi_G(K)$ is finite, as G is locally finite. Similarly as H is locally finite, then $\pi_H(K)$ is finite, where

$$\pi_H : G \times H \to H := (g, h) \mapsto h.$$

Hence it must be that K itself is finite, as $|K| \leq |\pi_G(K)||\pi_H(K)|$. Thus $G \times H$ is locally finite.

($\Leftarrow$) Given a subgroup G' of G, generated by a finite set $X \subseteq G'$, it is isomorphic to the subgroup $G' \times \{1_H\}$ of $G \times H$. $G' \times \{1_H\}$ is finitely generated by $X \times \{1_H\}$, whence the subgroup $G' \times \{1_H\}$ must be finite as $G \times H$ is locally finite. Hence G' must be finite, being in bijection with $G' \times \{1_H\}$. Thus G is locally finite. A similar argument applies to H. $\square$

Proposition 4.1.5. *Groups G and H are finitely generated if and only if $G \times H$ is finitely generated.*

Proof. ($\Rightarrow$) If X and Y are generating sets for G and H respectively, then the set $(X \times \{1_H\}) \cup (\{1_G\} \times Y)$ is a generating set for $G \times H$, which is finite.

($\Leftarrow$) If X is a finite generating set for $G \times H$, then the images $\pi_G(X)$ and $\pi_H(X)$ are finite generating sets for G and H respectively, where

$$\pi_G : G \times H \to G := (g, h) \mapsto g,$$
$$\pi_H : G \times H \to H := (g, h) \mapsto h.$$

$\square$

The theory of group presentations (or indeed, presentations more broadly in semigroup theory and universal algebra) has not been included in these lectures, and so we will state the following simply for noting, without proof.

Proposition 4.1.6. *Groups G and H are finitely presented if and only if $G \times H$ is finitely presented.* $\square$

Alongside these results come many other properties P, not necessarily finitary, for which the statement

Groups G and H have property P if and only if $G \times H$ has property P

holds. These include (and are not limited to) commutativity, nilpotency, solvability, and decidability of the word problem. In this sense, direct products of groups are particulary well behaved: we can construct new groups with any of these properties using the direct product of a family of groups we know that have them. Conversely, given a group with any of these properties, the factors of any direct representation must also have that property.

4.1.2 Finitary properties for direct products of semigroups

The question of properties P for which statements of the form

Semigroups S and T have property P if and only if $S \times T$ has property P

hold have been of interest in recent years, and have been comparably more elusive or harder to obtain than for groups. The reasoning is often as, unlike in the cases of groups or monoids, semigroups S and T are not necessarily present as isomorphic subsemigroups of the direct product $S \times T$.

For example, in the case of the proof of the statement for P being residual finiteness of two groups, the reverse implication was obtained only knowing that G and H could be respectively embedded in $G \times H$. For semigroups and residual finiteness, the above is indeed still true, but was only proved (for non-obvious reasons) as recently as 2009.

Theorem 4.1.7 (Gray, Ruškuc (2009) [6, Theorem 2])**.** *The direct product of semigroups S and T is residually finite if and only if S and T are residually finite.* $\square$

The same cannot be said for algebras in general, however.

Theorem 4.1.8 (Gray, Ruškuc (2009) [6, Example 1])**.** *Let A be the monounary algebra on $\mathbb{N}$ with operation $f(x) = x + 1$, and B be the monounary algebra on*

$\mathbb{N}$ *with operation*

$$g(x) = \begin{cases} x-1 & \text{if } x > 1 \\ 1 & \text{if } x = 1 \end{cases}$$

Then A is residually finite, $A \times B$ is residually finite, but B is not residually finite. □

Finite generation for the direct product of semigroups was indeed classified in terms of finite generation for the factors, but in this case we do not necessarily have a statement of the above form. The result is only true for infinite semigroups S which have no *indecomposable* elements. That is, $S^2 = S$, where

$$S^2 = \{ss' \in S : s, s' \in S\}.$$

Theorem 4.1.9 (Robertson, Ruškuc, Wiegold (1998) [8, Theorem 2.1])**.** *The direct product $S \times T$ of infinite semigroups S and T is finitely generated if and only if S and T are finitely generated, and $S^2 = S$, $T^2 = T$* □

In that same paper, the authors also gave the case for finite presentation of a direct product of semigroups, requiring the technical definition of "stable" which we will not cover.

Theorem 4.1.10 (Robertson, Ruškuc, Wiegold (1998) [8, Theorem 3.5])**.** *The direct product $S \times T$ of infinite semigroups S and T is finitely presented if and only if the following all hold:*

(1) $S^2 = S$, $T^2 = T$;

(2) *S and T are (finitely presented and) stable.* □

Turning to decision problems (asking whether or not an algorithm taking a finite set of information, and giving a yes or no answer to a specified question in finite time exists) for finite presentability now, we also have the following result.

Theorem 4.1.11 (I. Araújo, Ruškuc (2000) [1])**.** *Let S and T be semigroups, where S is finite in particular. Then it is decidable whether or not $S \times T$ is finitely presented.* □

The following is also still an open question, however.

Question 4.1.12. Is it possible to algorithmically determine if the direct product of two finitely presented semigroups (given by their finite presentations) is finitely presented?

4.2 Finitary properties in subdirect products

Comparatively to the previous section, we will see in this section that subdirect products have less tame behaviour than direct products with respect to preservation of finitary properties. That is, we will exhibit many counterexamples to the statement

A subdirect product C of algebras A and B has property P if and only if A and B have property P,

and indeed even counterexamples to both implication directions. We will start in the case of groups.

4.2.1 Finitary properties of subdirect products of groups

Even for "nice" infinite groups such as free groups, subdirect products have provided a tool for interesting counterexamples over the years. For example, in the case of finite presentation, the following is an example of a non-finitely generated subdirect product of two finitely presented groups.

Example 4.2.1 (Bridson, Miller (2009) [3, Example 3]). Let $A = \langle a_1, a_2 : \emptyset \rangle$ and
$B = \langle b_1, b_2 : \emptyset \rangle$ be two free groups of rank 2, and let

$$Q = \langle c_1, c_2 : q_1(c), q_2(c), \dots \rangle$$

be any two-generated group which is not finitely presented. Let $\varphi : A \to Q$ and $\psi : B \to Q$ be the unique epimorphisms extending the maps defined by $\varphi(a_i) = c_i$ and $\psi(b_i) = c_i$. The fiber product $\Pi(\varphi, \psi)$ of $A \times B$ is not finitely generated. $\triangle$

Further to this, in 1978 Grunewald gave examples of finitely generated subdirect products of two free groups of rank 2 which are not presented. The theorem giving these examples was extended by Baumslag & Roseblade in 1984, classifying the finitely presented subdirect products of free groups.

Theorem 4.2.2 (Baumslag, Roseblade (1984) [2, Theorem 2]). *Every finitely presented subgroup of a direct product of two free groups is a finite extension of a direct product of two free groups of finite rank.* $\square$

A final example we mention is due to Mikhaĭlova, who in 1966 gave a set of defining relations for a group which is a finitely generated subdirect product of two free groups, but has undecidable word problem.

4.2.2 Finitary properties of subdirect products of semigroups and algebras

The above examples are of course also counterexamples of semigroups and indeed of algebras. The study of finitary properties for subdirect products of semigroups and algebras however is a much more recent topic. For groups of course, every question involving subdirect products can be framed in terms of fiber products. We have the following results concerning finitary properties of fiber products.

Example 4.2.3 (Mayr, Ruškuc (2019) [7, Example 7.1])**.** We will see an example of a fiber product of two finitely presented monoids over a finite quotient, which is not finitely generated.

Let F_1 be the free monoid on a single generator a with identity 1, and let T be the two element monoid $\{0, 1\}$ under the standard multiplication. Define a homomorphism $\phi : F_1 \to \{0, 1\}$ by $a \mapsto 0$. The fiber product of F_1 and F_1 with respect to ϕ is not finitely generated, as any generating set for it must contain the pairs (a^i, a) for $i \in \mathbb{N}$, of which there are infinitely many. △

In this same work, the authors also exhibit a finitely presented subdirect product of two non-finitely presented monoids. They further give sufficient conditions for finite generation of a fiber product in the context of universal algebra. Note that A, B, C and D in the following theorem must all be algebras of the same type, and further that these algebras must belong to a *congruence permutable variety*, which heuristically is a class of algebras for which any pair of congruences commute under composition (such as the classes of groups and rings).

Theorem 4.2.4 (Mayr, Ruškuc (2019) [7, Example 7.1])**.** *If C is a fiber product of A and B over a common quotient D, and if A, B and D are finitely presented, then C is finitely generated.* □

Particular considerations for finitary properties of fiber products of free semigroups in particular have been given in [5].

4.3 Further open questions for subdirect products

We saw previously that fiber products and subdirect products do not coincide for semigroups, and hence not in general in universal algebra. They do coincide in varieties of algebras which are *congruence permutable* (where every pair of congruences commutes under composition). This limits our ability to construct examples of subdirect products of algebras with desirable properties. We hence give the following open problems.

Question 4.3.1. For a subdirect product of two algebras A and B which is not a fiber product, what fiber-like constructions can be used to obtain them? How do these constructions relate to the join relation on pairs of congruences?

Question 4.3.2. What finitary properties are preserved when taking subdirect products of algebras? Conversely, which properties of a subdirect product of two algebras will those algebras always inherit?

Question 4.3.3. How do the congruences on a direct product of semigroups (or algebras) relate to the congruences of the factors, and vice versa? What about the congruences on a subdirect product of semigroups (or algebras)?

Recommended further reading

To conclude these lecture notes, we will outline some of the recommended reading that covers the topics of semigroup theory and universal algebra we did not include, but alluded to in the lectures. We strongly encourage any reader interested in these topics to delve into this material, which is vast and particularly well written in our opinion. These are also followed by a bibliography of the papers covering the results of Chapter 4 in particular.

For more about the theory of semigroups, including on congruences, direct & subdirect products, we recommend the following:

- **J.M. Howie, "Fundamentals of Semigroup Theory". LMS Monographs. 12, The Clarendon Press, New York. (1995)**

  - Much more about the theory of semigroups in general can be found in this more recent monograph, than was given in these notes. In particular, it follows the notation we set out for congruences, and gives much more detail on generating congruences, congruence lattices, and the relation to the theory of free semigroups and semigroup presentations particularly well. It also has a very broad and celebrated overview on the algebraic theory of semigroups, including a rigorous treatment of "Green's relations" which we did not mention in these notes. Particular attention is also given to classes of semigroups including inverse semigroups, regular semigroups, clifford semigroups and also Rees' theorems on 0-simple semigroups. There are a number of results on direct and subdirect products, too, which we did not mention in these notes, as well as other related product constructions in the chapter on "semigroup amalgams".

- **A.H. Clifford, G.B. Preston, "The Algebraic Theory of Semigroups, Vol. 1". Surveys Amer. Math. Soc. 7. Providence, Rhode Island. (1961)**

  - One of the "original" monographs on semigroup theory, this Clifford & Preston work is another much cited and celebrated staple of the theory. In particular, it starts with a nice history of the theory of semigroups and how it came about, and then of course again covers many more of the preliminaries, further ideas and theorems from semigroup theory than were covered in these notes. Particular focus on representing semigroups in other ways (using matrices, and as unions of subsemigroups) is given in this first edition. There is also a much deeper second volume:

- **A.H. Clifford, G.B. Preston, "The Algebraic Theory of Semigroups, Vol. 2". Surveys Amer. Math. Soc. 7. Providence, Rhode Island. (1967).**

For more about the theory of Universal algebra, congruences on algebras, direct and subdirect products of algebras, we recommend the following:

- **S. Burris, H.P. Sankappanavar, "A course in universal algebra". Amer. Math. Monthly. 78, Springer-Verlag, New York. (1981)**

  - This monograph gives an excellent overview to get into universal algebra. Many concepts that you will be familiar with from algebra are generalised and exemplified in this book with a precise and well explained treatment, including the notions of subalgebras, homomorphisms, direct products, subdirect products, congruences, isomorphism theorems, and many many more. It also covers more specifically the theory of lattices, and connects the topics in the book to the related study of Model theory. In particular, the entire text is published online for free.

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