Homological finiteness properties for one-relator monoids and related monoids

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Outline

Background

Homological finiteness properties

Special monoids

Ancient history

- In 1932 Magnus solved the word problem for one-relator groups.
- Inspired by his results, people considered the word problem for other one-relator algebraic structures.
- In 1962, Shirshov solved the word problem for one-relator Lie algebras.
- The word problem for one-relator monoids remains open.
- The problem: given $M = \langle A \mid u = v \rangle$, decide whether two words over A represent the same element of M.
- There has been lots of work by people like Adjan, Lallement, Oganesyan, Guba, Howie and Pride.

What is known and why it is hard

- Adjan solved the word problem for cancellative one-relator monoids and those with defining relation $w = 1$ in the 60s.
- These cases reduce to Magnus's theorem.
- Most of the remaining results reduce the word problem from presentations with longer relations to shorter ones.
- Matiyasevich (1967) constructed a 2-generator, 3-relator monoid with undecidable word problem.
- Borisov gave a 12-relator group presentation with undecidable word problem based on Matiyasevich's example.
- Ivanov, Margolis and Meakin reduced the one-relator monoid word problem to the one-relator inverse monoid word problem with relation $w = 1$.
- Gray showed the word problem is undecidable for inverse monoids with defining relation $w = 1$.

Kobayashi's question

- When we can't solve a problem in math, we study variants of it.
- Kobayashi asked whether the word problem for one-relator monoids can be solved by a particularly nice algorithm.

Question (Kobayashi (2000))

Does every one-relator monoid admit a finite complete rewriting system?

- A complete rewriting system (CRS) is a presentation where you can solve the word problem as in the free group.
- Replacing left hand sides by right hand sides of a relation, will always result in a unique reduced word in finitely many steps.
- A finite CRS yields decidable word problem.

More on Kobayashi's question

- It is an open question if one-relator groups admit a finite CRS.
- It is an open question to decide whether a one-relator presentation is already complete.
- To prove that a monoid does not admit a finite CRS, we need invariants that can detect this.
- Homological finiteness is a popular such invariant.

Homological finiteness properties

- Homological finiteness properties for groups were introduced by Bieri in the 70s.
- The extension to monoids is straightforward and was studied in the 80s by:
    1. Bieri and Renz to introduce higher Σ-invariants of groups;
    2. Squier and Anick to study complete rewriting systems.
- Let M be a monoid and $\mathbb{Z}M$ its monoid ring.
- $\mathbb{Z}$ is the trivial module.
- M is of type FP_n with $0 \leq n \leq \infty$ if there is a free resolution

$$\cdots \to F_1 \to F_0 \to \mathbb{Z}$$

with F_i finitely generated for $0 \leq i \leq n$.

Warning: monoids are not ambidextrous

- The above definition is incomplete.
- We must specify if we use left or right modules.
- There are two distinct notions: left and right FP_n.
- Since the classes of monoids in this talk are left/right dual, we can get away with considering only left modules.
- The right hand versions then follow by duality.

Key facts

- Every finitely generated monoid is FP_1.
- Every finitely presented monoid is FP_2.
- Neither converse holds.
- Stallings gave the first finitely presented group that is not FP_3.
- Bieri gave finitely presented groups that are FP_n but not FP_{n+1} for all $n \geq 2$.

Theorem (Anick)

If M has a finite CRS, then M is FP_∞.

- This improves an earlier result of Squier for FP_3.
- To prove M has no finite CRS, it suffices to show M is not FP_n for some $n > 2$.

Lyndon's identity theorem

- Lyndon (1950) gave an explicit free resolution of $\mathbb{Z}$ for one-relator groups.
- An immediate consequence is the following theorem.

Theorem (Lyndon)

Let G be a one-relator group.

1. G *is* FP_∞.
2. $\mathrm{cd}(G) \leq 2$ *unless the relator is a proper power (i.e., G has torsion), in which case* $\mathrm{cd}(G) = \infty$.

- The cohomological dimension of a monoid is the length of a shortest free resolution of $\mathbb{Z}$.

Another question of Kobayashi

Question (Kobayashi (2000))

Is every one-relator monoid of type FP_∞?

- Anick's theorem provides the connection between this question and his previous one.
- Kobayashi (2000) proved one-relator monoids are FP_3.

Theorem (Gray, BS)

Every one-relator monoid is of type FP_∞.

- We have a fairly good, but still incomplete, understanding of cohomological dimension of one-relator monoids.

Topological methods

- Geometric group theorists use topology to establish homological finiteness properties.
- Wall approach: Construct an Eilenberg-Mac Lane space for G with appropriate finiteness properties.
- Brown approach: Find a 'nice' action of G on a contractible CW complex such that the cell stabilizers have appropriate finiteness properties.
- For monoids people typically establish homological finiteness properties by writing down explicit free resolutions.
- We introduce monoid analogues of both the Wall and Brown approaches.
- Our approach builds actions of one-relator monoids on contractible CW complexes whose associated cellular chain complexes provide resolutions of the trivial module.

Other tools

- We use Adjan-Oganesyan compression to work by induction on the size of the relator.
- We then must deal with incompressable one-relator presentations.
- One family was handled by Kobayashi.
- The rest of this talk is about the other base case.
- The inductive step will have to await another talk...

Special monoids

- A special monoid presentation is one of the form:

$$M = \langle A \mid w_1 = 1, \ldots, w_k = 1 \rangle.$$

- Any group is a special monoid.
- Any special monoid is either free or has non-trivial left/right invertible elements.
- So $\mathbb{N} \times \mathbb{N}$ is not special.
- For example, in $B = \langle a, b \mid ab = 1 \rangle$, a is right invertible and b is left invertible.
- Neither is invertible.

Results on special monoids

- Adjan (1960) proved the group of units of a special one-relator monoid is a one-relator group.
- He reduced the word problem to that of the group and invoked Magnus.
- Makanin (1966) proved the group of units G of a k-relator special monoid M is a k-relator group.
- He reduced the word problem of M to that of G.
- Zhang, in the 90s, gave an elegant approach to these results using infinite complete rewriting systems.
- He gave many structural results.
- He proved the monoid of right invertible elements of M is a free product of G with a finitely generated free monoid.

The main theorem

Theorem (Gray, BS)

Let M be a special monoid with group of units G.

1. *If G is FP_n, then M is FP_n.*
2. $\mathrm{cd}(G) \leq \mathrm{cd}(M) \leq \max\{2, \mathrm{cd}(G)\}$.

Corollary (Gray, BS)

Let M be a special one-relator monoid.

1. *M is FP_∞.*
2. $\mathrm{cd}(M) \leq 2$ *unless the relator is a proper power, in which case* $\mathrm{cd}(M) = \infty$.

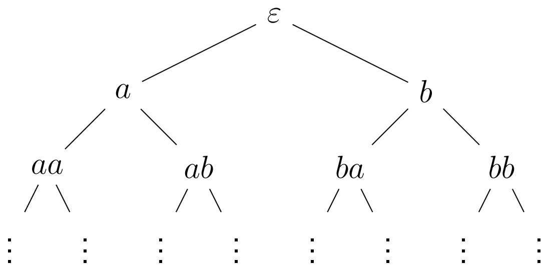
- Kobayashi obtained this for the case the relator is not a proper power.
- In general, homological properties of a monoid and its group of units are unrelated.

Idea of the proof

- We use the Brown approach to prove our main theorem.
- We construct an action of the special monoid on a tree.
- We use the chain complex of the tree to build a non-free resolution of $\mathbb{Z}$.
- We use Brown's method to replace the non-free resolution by one whose finiteness properties are controlled by G.

Cayley graph of a free monoid

$M = \{a, b\}^*$.



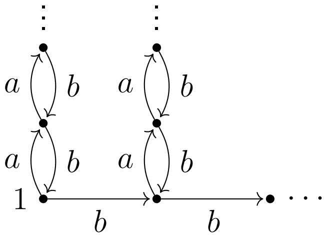
Free resolution:

$$0 \longrightarrow \mathbb{Z}M^2 \longrightarrow \mathbb{Z}M \longrightarrow \mathbb{Z} \longrightarrow 0$$

So M is FP_∞ and $\mathrm{cd}(M) = 1$.

Cayley graph of the bicyclic monoid

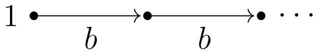
$B = \langle a, b \mid ab = 1 \rangle$.



All strong components isomorphic; each component has unique entrance; each vertex of the component is a unique right translate of the entrance by a right invertible element; the cone is identical from each entrance; contracting strong components yields a tree.

Tree of strong components for the bicyclic monoid

$B = \langle a, b \mid ab = 1 \rangle$.



- b acts by a right shift.
- a acts by a left shift but crushes the first edge to the vertex 1.

Geometry of special monoids

- Let M be a special monoid with Cayley graph Γ.
- The strong components of Γ are all isomorphic.
- Each strong component has a unique entrance (unique closest element to 1).
- The cone at each entrance is isomorphic to the whole Cayley graph.
- Each vertex of a strong component is a unique right translate of the entrance by a right invertible element.
- Identifying each strong component to a point yields a regular rooted tree $\mathcal{T}$.
- M acts on $\mathcal{T}$ by simplicial maps.
- The M-action might crush edges of $\mathcal{T}$.

FP_n for modules

- Let R be a ring.
- An R-module N is of type FP_n with $0 \leq n \leq \infty$ if there is a free resolution

$$\cdots \to F_1 \to F_0 \to N$$

with F_i finitely generated for $0 \leq i \leq n$.

Theorem (Bieri, Brown, Strebel)

If

$$\cdots \to N_1 \to N_0 \to N$$

is a resolution of N with N_i of type FP_{n-i} for $0 \leq i \leq n$, then N is FP_n.

The resolution from the tree $\mathcal{T}$

- Let M be a special monoid with group of units G.
- Assume that G is FP_n.
- Let R be the submonoid of right invertible elements.
- $R = G * F$ where F is a finitely generated free monoid.
- Let $\mathcal{T}$ be the tree of strong components of the Cayley graph Γ.
- We have a resolution from simplicial chain groups

$$0 \longrightarrow C_1(\mathcal{T}) \longrightarrow C_0(\mathcal{T}) \longrightarrow \mathbb{Z} \longrightarrow 0$$

- To get M is FP_n we need $C_0(\mathcal{T})$ is FP_n and $C_1(\mathcal{T})$ is FP_{n-1}.

The 0-chain group

- The vertices of $\mathcal{T}$ are the strong components of Γ.
- Each $m \in M$ can be uniquely written br where b is an entrance and $r \in R$.
- So $\mathbb{Z}M$ is a free right $\mathbb{Z}R$-module with basis the set of entrances (in bijection with strong components of Γ).
- Thus $\mathbb{Z}M \otimes_{\mathbb{Z}R} \mathbb{Z}$ is a free abelian group with basis in bijection with the strong components of Γ.
- The M-action on $\mathbb{Z}M \otimes_{\mathbb{Z}R} \mathbb{Z}$ is the action of M on the strong components under this identification.
- So $C_0(\mathcal{T}) \cong \mathbb{Z}M \otimes_{\mathbb{Z}R} \mathbb{Z}$.

Resolving the 0-chains

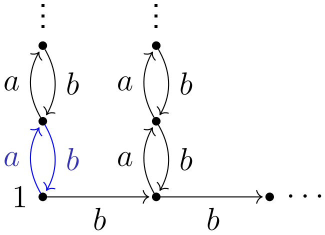
- Recall $C_0(\mathcal{T}) \cong \mathbb{Z}M \otimes_{\mathbb{Z}R} \mathbb{Z}$.
- The functor $\mathbb{Z}M \otimes_{\mathbb{Z}R} (-)$ is exact (since $\mathbb{Z}M$ is a free right $\mathbb{Z}R$-module) and sends $\mathbb{Z}R$ to $\mathbb{Z}M$.
- So if $F_\bullet \to \mathbb{Z}$ is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}R$, then $\mathbb{Z}M \otimes_{\mathbb{Z}R} F_\bullet$ is a free resolution of $\mathbb{Z}M \otimes_{\mathbb{Z}R} \mathbb{Z} \cong C_0(\mathcal{T})$.
- The rank of $\mathbb{Z}M \otimes_{\mathbb{Z}R} F_i$ is the same as the rank of F_i.
- So $C_0(\mathcal{T})$ is FP_n if R is FP_n.
- But $R = G * F$ with F a finitely generated free monoid.
- The class of FP_n monoids is closed under free product (Cremmens-Otto).
- Since G is FP_n and F is FP_∞, we have R is FP_n.

The 1-chain group

- The edges of $\mathcal{T}$ are the edges of Γ not belonging to a strong component.
- Let N be the free abelian group on the edges of Γ belonging to some strong component.
- Then $N \leq C_1(\Gamma)$ is a $\mathbb{Z}M$-submodule.
- $C_1(\mathcal{T}) \cong C_1(\Gamma)/N$.
- Note that $C_1(\Gamma)$ is a free $\mathbb{Z}M$-module with basis the edges $1 \xrightarrow{a} a$.
- The set E of edges of Γ is a free M-set.
- N has $\mathbb{Z}$-basis an M-invariant subset of E.
- We proved any invariant subset of a free M-set is free.
- So N is a free $\mathbb{Z}M$-module.
- We showed it is finitely generated.
- So $0 \longrightarrow N \longrightarrow C_1(\Gamma) \longrightarrow C_1(\mathcal{T}) \longrightarrow 0$ is a free resolution and hence $C_1(\mathcal{T})$ is FP_∞.

Cayley graph of the bicyclic monoid: revisited

$B = \langle a, b \mid ab = 1 \rangle$.



The blue edges freely generate the submodule of strong component edges.

Conclusion

- In summary, we have the resolution

$$0 \longrightarrow C_1(\mathcal{T}) \longrightarrow C_0(\mathcal{T}) \longrightarrow \mathbb{Z} \longrightarrow 0$$

- $C_0(\mathcal{T})$ is FP_n.
- $C_1(\mathcal{T})$ is FP_∞.
- So $\mathbb{Z}$ is FP_n by the Bieri-Brown-Strebel theorem.
- Thus M is of type FP_n.

The end

Thank you for your attention!