Min network of congruences on an inverse semigroup

Ying-Ying Feng

Foshan University, Guangdong, P. R. China

eNBSAN, 8th July, 2020

This is joint work with Li-Min Wang, Lu Zhang, Hai-Yuan Huang and Zhi-Yong Zhou.

Outline

1. Notations & terminologies
2. Congruence networks on inverse semigroups
3. Some future work

Various classes of semigroups

- group $\mathcal{SL} \subseteq \mathcal{I}$, $\mathcal{B} \cap \mathcal{I} = \mathcal{SL}$, $\cdots$
- regular semigroup — $(\forall a \in S)(\exists x \in S)\ axa = a$
- inverse semigroup — every element of S has a unique inverse

  — S is regular, and its idempotents commute
- completely regular semigroup

  — every element of S lies in a subgroup of S
- band — every element of S is idempotent
- semilattice — commutative idempotent semigroup
- Clifford semigroup — S is regular and the idempotents of S are central

  — a semilattice of groups
- E-unitary semigroup — $(\forall e \in E_S)(\forall s \in S)\ es \in E_S \Rightarrow s \in E_S$
- $\cdots$

Congruences

- congruence

  — a compatible equivalence relation

  $$(\forall s, t, s', t' \in S)\ [(s,t) \in \rho \text{ and } (s',t') \in \rho] \Rightarrow (ss', tt') \in \rho$$

  — both a left and a right congruence

  $$(\forall s, t, a \in S)\ (s,t) \in \rho \Rightarrow (as, at) \in \rho,\ (sa, ta) \in \rho$$

  - semigroup $S \xrightarrow{\text{congruence } \rho}$ quotient semigroup S/ρ
  - significance
    - obtain information on internal structure and homomorphic images
    - 'All the important structure theorems for inverse semigroups are based on various special congruences.'[1]

[1] Petrich, M.: *Inverse semigroups*. Wiley, New York (1984)

Congruences

- significance
  - obtain information on internal structure and homomorphic images
  - 'All the important structure theorems for inverse semigroups are based on various special congruences.'[2]

✓ S is an E-unitary inverse semigroup $\iff \sigma \cap \mathcal{L} = \varepsilon$

$S = \mathcal{M}(G, \mathcal{X}, \mathcal{Y}) = \{(A, g) \in \mathcal{Y} \times G \mid g^{-1}A \in \mathcal{Y}\}$

$\mathcal{Y} = S/\mathcal{L}$, $G = S/\sigma$

✓ S is a Clifford semigroup $\iff \mu = \eta$

$S = [Y; G_\alpha, \phi_{\alpha,\beta}]$

$Y = S/\eta = S/\mathcal{J}$

[2] Petrich, M.: *Inverse semigroups*. Wiley, New York (1984)

Congruences

- kernel–trace approach

  Let ρ be a congruence on S,

  $\operatorname{tr}\rho = \rho|_{E_S}, \qquad \ker\rho = \{x \in S \mid (\exists e \in E_S)\, x\,\rho\, e\}.$

Result

Let ρ be a congruence on S. Then

$$a\ \rho\ b \iff a^{-1}a\ \operatorname{tr}\rho\ b^{-1}b,\ ab^{-1} \in \ker\rho.$$

- $\mathcal{T}$, $\mathcal{K}$-relation

  Let $\rho, \theta \in \mathcal{C}(S)$,

  $\rho\,\mathcal{T}\,\theta \iff \operatorname{tr}\rho = \operatorname{tr}\theta, \qquad \rho\,\mathcal{K}\,\theta \iff \ker\rho = \ker\theta.$

Congruences

- kernel–trace approach

$$\operatorname{tr}\rho = \rho|_{E_S}, \qquad \ker\rho = \{x \in S \mid (\exists e \in E_S)\, x\,\rho\, e\}.$$

- $\mathcal{T}$, $\mathcal{K}$-relation

$$\rho\,\mathcal{T}\,\theta \iff \operatorname{tr}\rho = \operatorname{tr}\theta, \qquad \rho\,\mathcal{K}\,\theta \iff \ker\rho = \ker\theta.$$

Result

For any $\rho \in \mathcal{C}(S)$, $\rho\mathcal{T} = [\rho_t,\, \rho^T]$, $\rho\mathcal{K} = [\rho_k,\, \rho^K]$, where

$$a\,\rho_t\,b \iff ae = be \text{ for some } e \in E_S,\, e\,\rho\,a^{-1}a\,\rho\,b^{-1}b,$$

$$a\,\rho^T\,b \iff a^{-1}ea\,\rho\,b^{-1}eb \text{ for all } e \in E_S,$$

$$\rho_k = (\rho \cap \mathcal{L})^*,$$

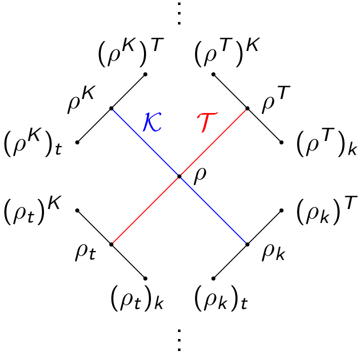
$$a\,\rho^K\,b \iff [xay \in \ker\rho \iff xby \in \ker\rho \text{ for all } x, y \in S^1].$$

Congruences

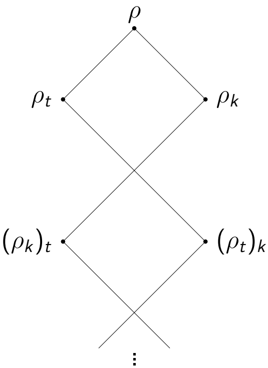
- kernel–trace approach
- $\mathcal{T}$, $\mathcal{K}$-relation
- congruence networks
  - single out various classes of semigroups of particular interest
  - structure

Congruence network

$$\mathcal{T} \cap \mathcal{K} = \varepsilon$$



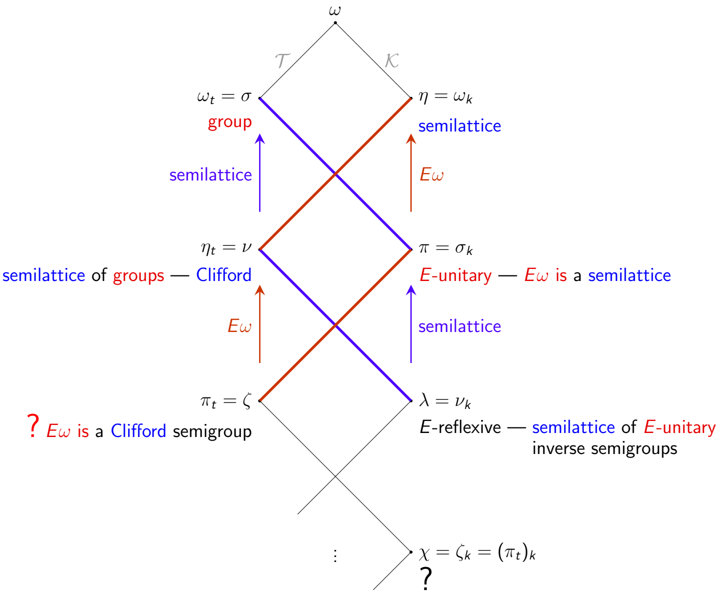
congruence network of ρ



min network of ρ

Min network of ω on inverse semigroups

[1982, Petrich – Reilly]



min network of ω

Proposition

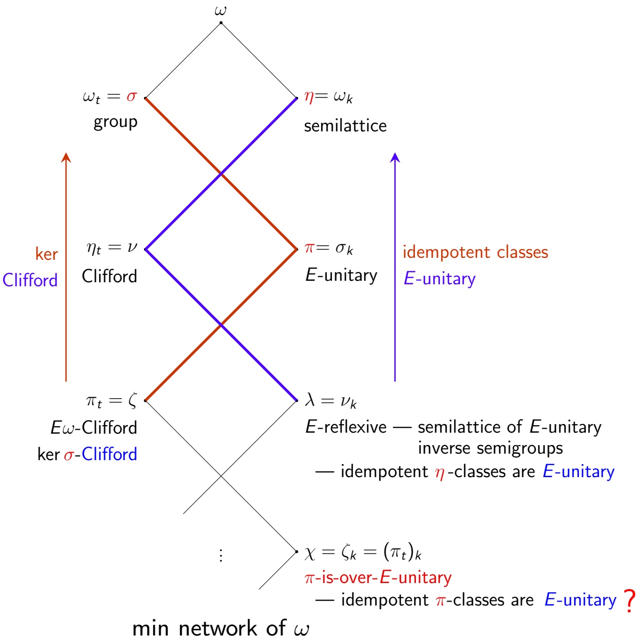
The following conditions on an inverse semigroup S are equivalent.
(1) *S is an $E\omega$-Clifford semigroup;*
(2) $\sigma \cap \mathcal{L}$ *is a congruence;*
(3) $\sigma \cap \mathcal{R}$ *is a congruence;*
(4) $\sigma \cap \mathcal{L} = \sigma \cap \mathcal{R}$;
(5) $\sigma \cap \mathcal{L} = \sigma \cap \mu$;
(6) *there exists an idempotent separating E-unitary congruence on S;*
(7) $\pi \subseteq \mu$;
(8) $\pi_t = \varepsilon$;
(9) *$e\sigma$ is a Clifford semigroup for every $e \in E(S)$;*
(10) *S satisfies the implication $xy = x \Rightarrow y \in E(S)\zeta$;*
(11) $E(S)\omega \subseteq E(S)\zeta$;
(12) $\pi \cap \mathcal{F} = \varepsilon$.

Proposition

The following statements concerning a congruence ρ on an inverse semigroup S are equivalent.
(1) *ρ is an $E\omega$-Clifford congruence;*
(2) $\pi_\rho \subseteq \rho^T$, *where π_ρ is the least E-unitary congruence on S containing ρ;*
(3) $\operatorname{tr} \pi_\rho = \operatorname{tr} \rho$.

- Wang, L. M., Feng, Y. Y.: *$E\omega$*-Clifford congruences and *$E\omega$-E*-reflexive congruences on an inverse semigroup. Semigroup Forum **82**, 354–366 (2011)

Min network of ω on inverse semigroups



- Feng, Y. Y., Wang, L. M., Zhang, L., Huang, H. Y.: A new approach to a network of congruences on an inverse semigroup. Semigroup Forum **99**, 465–480 (2019)

Min network of ω on inverse semigroups

Definition

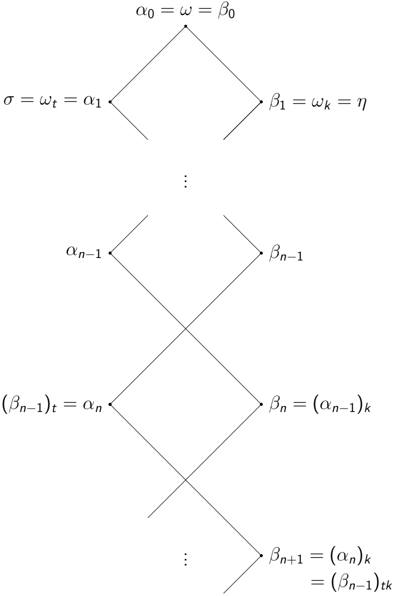
On S we define inductively the following two sequences of congruences:

$$\alpha_0 = \omega = \beta_0;$$

$$\alpha_n = (\beta_{n-1})_t, \quad \beta_n = (\alpha_{n-1})_k,$$

$$\text{for } n \geqslant 1.$$

We call the aggregate $\{\alpha_n, \beta_n\}_{n=0}^{\infty}$, together with the inclusion relation for congruences, the min network of ω on S.



min network of ω

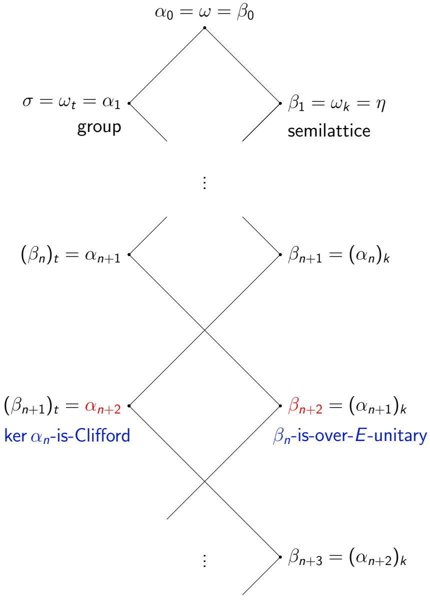
Min network of ω on inverse semigroups

Definition

An inverse semigroup for which $\ker \alpha_n$ is a Clifford semigroup is called a $\ker \alpha_n$-is-Clifford semigroup. An inverse semigroup S is called a β_n-is-over-E-unitary semigroup if $e\beta_n$ is E-unitary for each $e \in E_S$.

Theorem

(1) α_{n+2} *is the least* $\ker \alpha_n$*-Clifford congruence on* S*;*
(2) β_{n+2} *is the least* β_n*-is-over-*E*-unitary congruence on* S*.*



min network of ω

ker α_n-is-Clifford semigroups and β_n-is-over-E-unitary semigroups

Proposition

For $n \geqslant 1$, the following conditions on an inverse semigroup S are equivalent:

(1) S *is a* ker α_n*-is-Clifford semigroup;*

(2) $[a \, \alpha_n \, b$ *and* $a^{-1}a \leq b^{-1}b] \implies aa^{-1} \leq bb^{-1}$;

(3) $\alpha_n \cap \mathcal{L} = \alpha_n \cap \mathcal{R}$;

(4) $\alpha_n \cap \mathcal{L}$ *is a congruence;*

(5) $\alpha_n \cap \mathcal{R}$ *is a congruence;*

(6) $\alpha_n \cap \mathcal{L} = \alpha_n \cap \mu$;

(7) *there exists an idempotent separating* β_{n-1}*-is-over-*E*-unitary congruence on* S;

(8) $\beta_{n+1} \subseteq \mu$; (9) $(\beta_{n+1})_t = \varepsilon$;

(10) $\beta_{n+1} \cap \mathcal{F} = \varepsilon$; (11) $\ker \alpha_n \subseteq E_S\zeta$;

(12) S *satisfies the implication* $xy = x$, $x^{-1}x \, \alpha_n \, yy^{-1} \Rightarrow y \in E_S\zeta$.

Proposition

For $n \geqslant 1$, the following conditions on an inverse semigroup S are equivalent:

(1) S *is a* β_n*-is-over-*E*-unitary semigroup;*

(2) $\beta_n \cap \mathcal{F}$ *is a congruence;*

(3) $\beta_n \cap \mathcal{C}$ *is a congruence;*

(4) $\beta_n \cap \mathcal{F} = \beta_n \cap \tau$;

(5) $\beta_n \cap \mathcal{C} = \beta_n \cap \tau$;

(6) *there exists an idempotent pure* ker α_{n-1}*-is-Clifford congruence on* S;

(7) $\alpha_{n+1} \subseteq \tau$; (8) $\alpha_{n+1} \cap \mathcal{L} = \varepsilon$;

(9) $(\alpha_{n+1})_k = \varepsilon$; (10) $\operatorname{tr} \beta_n \subseteq \operatorname{tr} \tau$;

(11) S *satisfies the implication* $xy = x$, $x^{-1}x \, \alpha_{n+1} \, yy^{-1} \Rightarrow y \in E_S$.

$\ker \alpha_n$-is-Clifford congruences and β_n-is-over-E-unitary congruences

Proposition

For $n \geqslant 1$, the following statements concerning a congruence ρ on an inverse semigroup S are equivalent:
(1) ρ is a $\ker \alpha_n$-is-Clifford congruence;
(2) $(\beta_{n+1})_\rho \subseteq \rho^T$, where $(\beta_{n+1})_\rho$ is the least β_{n-1}-is-over-E unitary congruence on S containing ρ;
(3) $\operatorname{tr}(\beta_{n+1})_\rho = \operatorname{tr}\rho$.

Proposition

For $n \geqslant 1$, the following statements concerning a congruence ρ on an inverse semigroup S are equivalent:
(1) ρ is a β_n-is-over-E-unitary congruence;
(2) $(\alpha_{n+1})_\rho \subseteq \rho^K$, where $(\alpha_{n+1})_\rho$ is the least $\ker \alpha_{n-1}$-is-Clifford congruence on S containing ρ;
(3) $\ker(\alpha_{n+1})_\rho = \ker \rho$.

Theorem

α_{n+2} is the least $\ker \alpha_n$-Clifford congruence on S.

Theorem

β_{n+2} is the least β_n-is-over-E-unitary congruence on S.

Quasivarieties

Definition (Petrich - Reilly, 1982)

An inverse semigroup S might satisfy one of the following implications:

(A_0) $x = y$; (A_1) $x^{-1}x = y^{-1}y$;

(A_2) $y \in E\zeta$;

(A_n) $xy = x,\ x\,\beta_{n-3}\,y \Rightarrow y \in E\zeta,\ n \geqslant 3$;

(B_0) $x = y$; (B_1) $y \in E$;

(B_n) $xy = x,\ x\,\beta_{n-2}\,y \Rightarrow y \in E,\ n \geqslant 2$.

Definition

An inverse semigroup S might satisfy one of the following implications:

(A'_0) $x = y$; (A'_1) $x^{-1}x = y^{-1}y$;

(A'_2) $y \in E\zeta$;

(A'_n) $xy = x,\ x^{-1}x\,\alpha_{n-2}\,yy^{-1} \Rightarrow y \in E\zeta,\ n \geqslant 3$;

(B'_0) $x = y$; (B'_1) $y \in E$;

(B'_n) $xy = x,\ x^{-1}x\,\alpha_{n-1}\,yy^{-1} \Rightarrow y \in E,\ n \geqslant 2$.

Theorem (Petrich - Reilly, 1982)

(1) α_n *is the minimum congruence* ρ *on* S *such that* S/ρ *satisfies* (A_n);

(2) β_n *is the minimum congruence* ρ *on* S *such that* S/ρ *satisfies* (B_n).

Theorem

(1) α_n *is the minimum congruence* ρ *on* S *such that* S/ρ *satisfies* (A'_n);

(2) β_n *is the minimum congruence* ρ *on* S *such that* S/ρ *satisfies* (B'_n).

Min network of ω on inverse semigroups

Theorem

Let n be a non-negative integer. The following statements are valid in any inverse semigroup S.

(1) *Every η-class of S/β_{2n+3} is a β_{2n}-is-over-E-unitary semigroup;*

(2) *every η-class of $S/\alpha_{2(n+2)}$ is a* ker α_{2n+1}*-is-Clifford semigroup;*

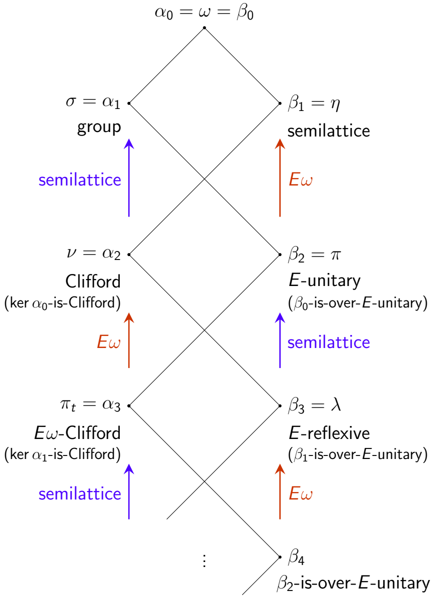
(3) $(E_{S/\alpha_{2n+3}})\omega$ *is a* ker α_{2n}*-is-Clifford semigroup;*

(4) $(E_{S/\beta_{2(n+2)}})\omega$ *is a β_{2n+1}-is-over-E-unitary semigroup.*

Theorem

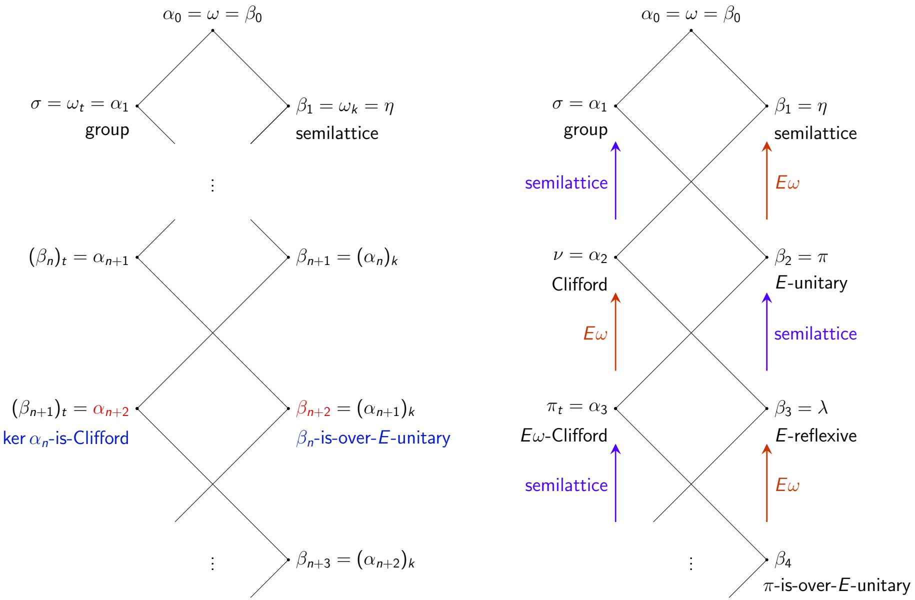
(1) α_{n+2} *is the least* ker α_n*-Clifford congruence on S;*

(2) β_{n+2} *is the least β_n-is-over-E-unitary congruence on S.*



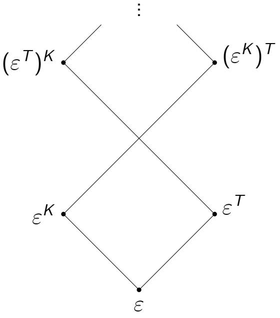
min network of ω

Min network of ω on inverse semigroups

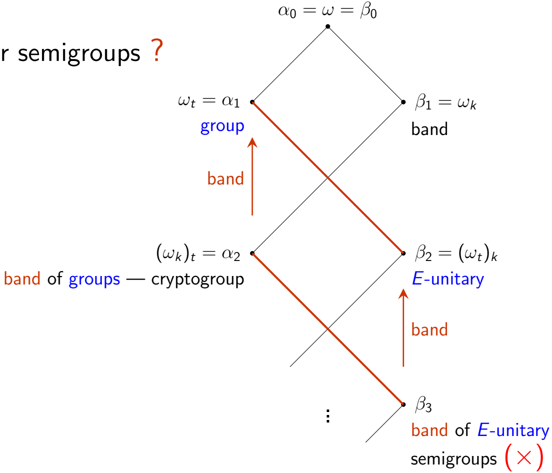


Some future work

- Pattern suitable for others ?
  - In general, NO!
  - Completely regular semigroups ?
- Max network of ε ?



max network of ε



min network of ω on regular semigroups

Bibliography

Billhardt, B.:
On inverse semigroups the closure of whose set of idempotents is a Clifford semigroup.
Semigroup Forum **44**, 320–331 (1992)

Pastijn, F., Petrich, M.:
The congruence lattice of a regular semigroup.
J. Pure Appl. Algebra **53**, 93–123 (1988)

Petrich, M.:
Inverse Semigroups.
Wiley, New York (1984)

Petrich, M.:
Congruences on Inverse Semigroups.
J. Algebra **55**, 231–356 (1978)

Petrich, M., Reilly, N. R.:
A network of congruences on an inverse semigroup.
Trans. Amer. Math. Soc. **270**, 309–325 (1982)

Thank you !