Towards a modular version of Klyachko's theorem on the powers.

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(joint work with Roger Bryant)
Free associative algebra

Let $V$ be a f.d. vector space over a field $K$ with basis $\{x_1, \ldots, x_n\}$. Let $T(V)$ be the free associative $K$-algebra freely generated by $x_1, \ldots, x_n$.

Thus

$$T(V) = T^0(V) \oplus T^1(V) \oplus \cdots \oplus T^r(V) \oplus \cdots$$

where $T^0(V) = K \cdot 1$ and $T^r(V)$ is a f.d. subspace with basis $\{x_{i_1} \cdots x_{i_r} : i_1, \ldots, i_r \in \{1, \ldots, n\}\}$.

- $T^r(V)$ is spanned by all $v_1 \cdots v_r$ with $v_1, \ldots, v_r \in V$.
- $T^r(V)$ may be identified with $V^\otimes r$ via $v_1 \cdots v_r \leftrightarrow v_1 \otimes \cdots \otimes v_r$.
Free Lie algebra

$T(V)$ is a Lie algebra under $[\cdot, \cdot]$, where

$[a, b] := ab - ba$

for all $a, b \in T(V)$.

Let $L(V)$ denote the Lie subalgebra of $T(V)$ generated by $x_1, \ldots, x_n$.

Then, by a theorem of Witt, $L(V)$ is a free Lie algebra freely generated by $x_1, \ldots, x_n$.

$L(V) = L'(V) \oplus L^2(V) \oplus \cdots \oplus L^r(V) \oplus \cdots$

where $L^r(V) = L(V) \cap T^r(V)$.

- $L'(V)$ is identified with $V$
- $L^r(V)$ is spanned by all left-normed commutators
  
  $[v_1, v_2, \ldots, v_r]$

  with $v_1, v_2, \ldots, v_r \in V$. 
Tensor and lie powers

Now let $G$ be a group and suppose that $V$ is a $kG$-module.

- $T^r(V) \ (= V \otimes_r)$ is a $kG$-module called the $r$th tensor power of $V$.

**Action of $g \in G$:**

$$(v_1, v_2, \ldots, v_r) g = (v_1g)(v_2g) \cdots (v_r g)$$

i.e.

$$(v_1 \otimes v_2 \otimes \cdots \otimes v_r) g = (v_1g) \otimes (v_2g) \otimes \cdots \otimes (v_r g)$$

- $L^r(V)$ is a $kG$-submodule of $V \otimes_r$ called the $r$th lie power of $V$.

**Action of $g \in G$:**

$$[v_1, v_2, \ldots, v_r] g = [v_1g, v_2g, \ldots, v_r g]$$
Tensor and lie powers

We started with a finite dimensional $K$-module $V$ and created two (infinite) families of finite dimensional $K$-modules:

Tensor powers: $V^\otimes 0, V^\otimes 1, V^\otimes 2, \ldots, V^\otimes r, \ldots$

Lie powers: $L^1(V), L^2(V), \ldots, L^r(V), \ldots$

Research theme

Given a field $K$, a group $G$ and a particular $K$-module $V$, find $V^\otimes r, L^r(V)$ (and related modules) up to isomorphism.

Which indecomposable modules occur as direct summands?
With what multiplicity?
Polynomial $GL(n, K)$-modules

From now on we take $K$ to be an algebraically closed field of characteristic $p > 0$, $G = GL(n, K)$.

- $W$ a finite-dimensional $kG$-module with basis $\{w_1, \ldots, w_d\}$. For each $g = (g_{ij}) \in G$, let $A_g$ denote the $d \times d$ matrix giving the action of $g$ on $W$ w.r.t. the basis $\{w_1, \ldots, w_d\}$.

- We say that $W$ is a polynomial module (of degree $r$) if there exist $d^2$ polynomials (homogeneous of degree $r$) $\phi_{k,l}$ in $n^2$ variables such that the $(k,l)^{th}$ entry of $A_g$ is $\phi_{k,l}(g_{ij})$ for all $g \in G$. 

Polynomial $GL(n, k)$-modules

Let $\text{mod}_k(n, r)$ denote the category of finite dimensional polynomial $kG$-modules of degree $r$.

**FACTS**

1. If $W$ is polynomial of degree $r$ and $U$ is polynomial of degree $s$ then $W \otimes_k U$ is polynomial of degree $rs$.

2. $\text{mod}_k(n, r)$ is closed under direct sums, submodules and factor modules.

**Examples**

1. Let $E$ denote the $n$-dimensional natural $kG$-module. Then $E$ is a polynomial module of degree 1.

2. $E \otimes r$ is a polynomial module of degree $r$, by FACT 1

3. $L^r(E)$ is a polynomial module of degree $r$, by FACT 2
The Schur functor

let \( \text{mod}(K_{Sr}) \) denote the category of all f.d. \( K_{Sr} \)-modules.

If \( n > r \) then there is an exact functor

\[
fr : \text{mod}_K(n, r) \to \text{mod}(K_{Sr})
\]

called the Schur functor.

- \( fr(W) = W^{(r)} \)

(a certain subspace (weight-space) of \( W \), regarded as a \( K_{Sr} \)-module via the embedding \( Sr \subseteq GL(n, K) \)

\[
\sigma \mapsto \left( \begin{array}{c|c} \text{perm} & 0 \\ \hline 0 & \text{id} \end{array} \right)
\]

- \( fr(E^\otimes r) = K_{Sr} \)

- \( fr(L^r(E)) = \text{lie}(r) \)

\( \uparrow \)

\text{lie module of the symmetric group.}
Indecomposable summands of $E_R$

- For each partition $\lambda$ of $\lambda$ into at most $n$ parts there is a certain indecomposable "tilting" module $T(\lambda)$ in $\text{mod}_k(n,r)$.

- We say that a partition $\lambda$ is row $p$-regular if no $p$ parts of $\lambda$ are equal.

- Let $n \geq r$. Then the set

$$\{f_r(T(\lambda)) : \lambda \text{ is row } p\text{-regular} \}$$

is a full set of projective indecomposable $K_{Sr}$-modules.

- Write $p^\lambda = f_r(T(\lambda))$. The head of $p^\lambda$ is a simple $K_{Sr}$-module denoted by $D^\lambda$. 
Indecomposable summands of $E^{\otimes r}$

$$E^{\otimes r} \cong \bigoplus t_a T(a)$$

Where the sum ranges over all row $p$-regular partitions of $r$ into at most $n$ parts and $t_a = \dim D^a$.

Remarks

1. When $\text{char } k = p = 0$, this result is classical.

The modules $E^{\otimes r}$ are known to be semisimple (Schur, 1927) and we have

- $T(a)$ is simple
- $D^a = S^a$, the Specht module

so that $t_a = \frac{n!}{T(\text{hook lengths})} \binom{\# \text{ standard tableaux of shape } a}{S^a}$

2. If $\text{char } k = p > 0$ then there is not, in general, an explicit formula for the multiplicities $t_a$. 
Indecomposable summands of $L^r(E)$

- Recall that $L^r(E)$ is a $K_A$-submodule of $E^\otimes r$. We would like to describe $L^r(E)$ up to isomorphism.

**Useful fact** If $\text{char} = p^\infty$, then $L^r(E)$ is a direct summand of $E^\otimes r$.

- Thus, in this case,

$$L^r(E) \cong \bigoplus_l L_a T(a)$$

where the sum ranges over all row $p$-regular partitions of $r$ into at most $n$ parts and $0 \leq l_a \leq t_a = \dim D_a$.

**Question:** What is $L_a$?
Indecomposable summands of $L^r(E)$

If $\text{char } K = p + r$ then $L^r(E)$ is a direct summand of $E^\otimes r$ and we let $l_\lambda$ denote the multiplicity of $T(\lambda)$ in $L^r(E)$.

**Multiplicity formulae**

- If $\text{char } K = 0$
  \[ l_\lambda = \frac{1}{r} \sum \mu(d) X^d(\tau^r \lambda) \quad \text{Wever, 1949} \]

- If $\text{char } K > 0$
  \[ l_\lambda = \frac{1}{r} \sum \mu(d) \beta^d(\tau^r \lambda) \quad \text{Donkoh + Erdmann, 1998} \]

Here $\mu$ denotes the Möbius function,

- $X^d$ denotes the character of $D^2 = S^2$

- $\beta^d$ denotes the Brauer character of $D^2$

and $\tau$ denotes a cycle of length $r$ in $S_r$

**Question:** When is $l_\lambda > 0$?
Klyachko's theorem

Let \( \text{char } k = 0 \).

Recall that

\[ E \otimes r \cong \bigoplus t_\lambda \, T(\lambda) \]

where the sum ranges over all partitions of \( r \) into at most \( n \) parts, the modules \( T(\lambda) \) are simple and \( t_\lambda = \dim S^\lambda \), the Specht module.

\[ \text{Theorem } (\text{Klyachko, 1974}) \]

Let \( r \geq 3 \) and let \( \lambda \) be a partition of \( r \) into at most \( n \) parts.

\[ T(\lambda) \mid L^r(CE) \iff \lambda \neq (1^r) \]
\[ \lambda \neq (r) \]
\[ \lambda \neq (2^2) \]
\[ \lambda \neq (2^3) \]

That is, almost all isomorphism types of indecomposable summands of \( E \otimes r \) occur as summands of \( L^r(CE) \).
Klyachko's theorem - Sketch of proof

- $S_r$ acts on the left of $E^{\otimes r}$ by place permutations.

- Let $\xi$ be a primitive $r$th root of unity in $K$ and let

$$U_\xi = \{ u \in E^{\otimes r} : \tau u = \xi u \}$$

where $\tau$ is a cycle of length $r$ in $S_r$.

Note: $U_\xi$ is the module from Roger's talk in the case where we take $G = GL(n, K)$ and $V = E$ the natural $K[G]$-module.

First show that $L^r(E) \cong U_\xi$ as $K[G]$-modules for any primitive root $\xi$.

- Thus we want to know when $T(\mathcal{A}) | U_\xi$.

- Since $T(\mathcal{A})$ is simple we find that

$$fr(T(\mathcal{A})) = D^r$$

- Also,

$$fr(U_\xi) = \mathcal{I} \uparrow^{S_r}_{S_2}$$

where $\mathcal{I}$ is any 1-dimensional faithful $K[G]$-module.
Klyachko's theorem - Sketch of proof

• Thus

\[ T(\lambda) \mid L^\infty(E) \iff D^3 \mid I \uparrow_{<C^5} \]

\[ \iff \text{Hom} \left( I \uparrow_{<C^5}, D^3 \right) \neq 0 \]

\[ \iff \text{Hom} \left( I, D^3 \downarrow_{<C^5} \right) \neq 0. \]

Now show that if \( r > 6 \) and \( D^3 \) is faithful, then \( D^3 \downarrow_{<C^5} \) contains a faithful 1-dimensional submodule.

• Remains to show that \( T(1^r), T(r) \uparrow L^\infty(E) \)

and \( T(2^3) \uparrow L^4(E), T(2^3) \uparrow L^6(E) \).

Use multiplicity formulae, for example.

\[ \square \]

Remark: If \( \text{char} k = p > 0 \) and \( p \nmid r \), then this argument works with little modification so that almost all isomorphism types of indecomposable summands of \( E \otimes r \) occur as summands of \( L^\infty(E) \).
Klyachko's theorem in arbitrary degree?

**Question** Which indecomposable summands $T(A)$ of $E^\otimes r$ also occur as direct summands of $L^r(E)$?

- If $\text{char} \ k \mid r$: almost all.
- What about when $\text{char} \ k \nmid r$?

Suppose that $r = p^m k$ where $p + k$. Then the Decomposition Theorem gives

$$L^{p^m k}(E) \cong L^{p^m}(B_{r_k}) \oplus L^{p^{m-1}}(B_{p k}) \oplus \ldots \oplus L^1(B_{p^m k})$$

where $B_{p^i k} \in L^{p^i k}(E)$ and $B_{p^i k} \mid E^\otimes p^i k$.

- If $k = 1$, this decomposition is trivial.
- For $k > 1$, we find that $B_{p^m k} \mid L^{p^m k}(E)$

Idea: Try to show that almost all $T(A)$ occur as summands of $B_{p^m k}$. 
A modular version of Kuyachko's Theorem?

We will use information about the B's from Roger's talk:

Let $\Delta$ be the set of $k^m$ roots of unity in $K$.

1. $B_{p^m k}$ is isomorphic to a direct sum of modules of the form $U_{s_1} \otimes \cdots \otimes U_{s_{p^m}}$ where $s_1, \ldots, s_{p^m} \in \Delta$ and $s_1, \ldots, s_{p^m}$ is a primitive $k^m$ root of unity.

   Recall: $U_s = \{ u \in E^{\otimes k} : \tau u = su \}$ where $\tau$ is a cycle of length $r$ in $S_r$.

2. Suppose that $k \geq 2$ and $p^m \neq 2$ if $k = 3$. Then each such product $U_{s_1} \otimes \cdots \otimes U_{s_{p^m}}$ occurs as a summand of $B_{p^m k}$.
Main result

let \( r = p^{mk} \) where \( m \geq 0, k \geq 2 \) and \( p + k \).

let \( \lambda \) be a \( p \)-regular partition of \( r \) into at most \( n \) parts.

\[ T(\lambda) \mid Bp^{mk} \iff \begin{cases} 
(i) \lambda \neq \mu', & \text{where } \mu'(=p^{1,-p^{1},b}) \text{ with } 1 \leq b \leq p-1. \\
(ii) \lambda \neq (r) \\
(iii) \lambda \neq (2^{2}) \\
(iv) \lambda \neq (2^{3}) \\
v) \lambda \neq (4,2) \text{ if } p^m = 2 \text{ and } k = 3.
\end{cases} \]

Thus almost all isomorphism types of indecomposable summands of \( E \otimes r \) occur as summands of \( Bp^{mk} \) and hence of \( L^r(E) \).

Remarks

* Proof is an extended version of Klyachko's argument.

[We show that \( T(\lambda) \mid U_8, \otimes \ldots \otimes U_8^{pm} \) for some choice of \( s_1, \ldots, s_{pm} \) with \( s_1, \ldots, s_{pm} \) primitive.]

* Need \( k \geq 2 \) so that we have "enough choice."