Adams operations on the Green ring of a finite group

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Shameless self-promotion

 Joint work with Professor Roger Bryant.
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Let $K$ be a field of prime characteristic $p$ and let $G$ be a finite group. We consider finite-dimensional right $KG$-modules.

Notice that the one-dimensional module on which $G$ acts trivially is the identity element in $R_{KG}$. Thus $K = 1$ in $R_{KG}$. 
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$KG$-modules: $U \oplus V \quad U \otimes K V \quad V^\otimes n$

Elements of $R_{KG}$: $U + V \quad UV \quad V^n$
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Notice that the one-dimensional module on which $G$ acts trivially is the identity element in $R_{KG}$. Thus $K = 1$ in $R_{KG}$. 


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For $r = 1, \ldots, q$ write $V_r = KG/KG(g - 1)^r$. Then $V_r$ is indecomposable of dimension $r$ and hence $R_{KG}$ has $\mathbb{Z}$-basis $\{V_1, \ldots, V_q\}$. 

The Green ring of a cyclic $p$-group
Let $G = \langle g \rangle$ be a cyclic $p$-group of order $q$. There are $q$ indecomposable $KG$-modules up to isomorphism. For $r = 1, \ldots, q$ write $V_r = KG/KG(g - 1)^r$. Then $V_r$ is indecomposable of dimension $r$ and hence $R_{KG}$ has $\mathbb{Z}$-basis $\{V_1, \ldots, V_q\}$.

Each indecomposable $V_r$ has basis $\{y_1, \ldots, y_r\}$ and the action of $g$ on $V_r$ with respect to this basis is given by the Jordan block

$$
\begin{pmatrix}
1 & 1 \\
& & \ddots & 1 \\
& & & 1 & 1 \\
& & & 1 & 1
\end{pmatrix}.
$$

(Notice that $V_1$ is the one-dimensional trivial module and $V_q$ is the regular $KC$-module.)
Let $V$ be a vector space over $K$ with basis $\{x_1, \ldots, x_r\}$. Write

$S(V) = K[x_1, \ldots, x_r]$ (free associative commutative $K$-algebra),

$\Lambda(V) =$ free associative $K$-algebra on $x_1, \ldots, x_r$ subject to

\[ x_i \wedge x_i = 0 \text{ and } x_i \wedge x_j = -x_j \wedge x_i. \]
Symmetric and exterior powers

Let $V$ be a vector space over $K$ with basis $\{x_1, \ldots, x_r\}$. Write

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x_i \wedge x_i = 0 \text{ and } x_i \wedge x_j = -x_j \wedge x_i.
\]

Take decompositions into homogeneous components:

\[
S(V) = S^0(V) \oplus S^1(V) \oplus \cdots \oplus S^n(V) \oplus \cdots,
\]

\[
\Lambda(V) = \Lambda^0(V) \oplus \Lambda^1(V) \oplus \cdots \oplus \Lambda^n(V) \oplus \cdots
\]

These components are the **symmetric powers** and **exterior powers** of $V$. 
Symmetric and exterior powers

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These components are the **symmetric powers** and **exterior powers** of $V$.

If $V$ is a $KG$ module then $S^n(V)$ and $\Lambda^n(V)$ become $KG$-modules by linear substitutions.
Properties of symmetric and exterior powers

The $n$th symmetric power $S^n(V)$ has $K$-basis

$$\{x_{i_1}x_{i_2}\cdots x_{i_n} : 1 \leq i_1 \leq i_2 \leq \cdots \leq i_n \leq r\}.$$ 

The $n$th exterior power $\Lambda^n(V)$ has $K$-basis

$$\{x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_n} : 1 \leq i_1 < i_2 < \cdots < i_n \leq r\}.$$ 

Thus $\dim S^n(V) = \binom{n+r-1}{n}$ and $\dim \Lambda^n(V) = \binom{r}{n}$. 
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\{x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_n} : 1 \leq i_1 < i_2 < \cdots < i_n \leq r\}.
\]

Thus \( \dim S^n(V) = \binom{n+r-1}{n} \) and \( \dim \Lambda^n(V) = \binom{r}{n} \).

It is also easy to check that

\[
S^n(U \oplus V) \cong \bigoplus_{a+b=n} S^a(U) \otimes S^b(V)
\]

and

\[
\Lambda^n(U \oplus V) \cong \bigoplus_{a+b=n} \Lambda^a(U) \otimes \Lambda^b(V).
\]
We started with a finite-dimensional $KG$-module $V$ and have created two families of $KG$-modules. What can we say about these new modules?

**Problem.** Determine $S^n(V)$ and $\Lambda^n(V)$ up to isomorphism, i.e. as elements of $R_{KG}$.
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**Examples.**

$S^0(V) \cong \Lambda^0(V) \cong K$, written as 1 in $R_{KG}$.

$S^1(V) \cong \Lambda^1(V) \cong V$.

Note that $S^n(V) \not\cong \Lambda^n(V)$ for $n > 1$, by dimensions.

In particular, $\Lambda^n(V) = 0$ for $n > r$, whilst $S^n(V) \neq 0$ for all $n$. 
Adams operations

Consider the power series ring \((\mathbb{Q} \otimes R_{KG})[[t]]\).
Define \(\psi^n_S(V)\) and \(\psi^n_\Lambda(V)\) in \(\mathbb{Q} \otimes R_{KG}\) by

\[
\psi^1_S(V)t + \frac{1}{2} \psi^2_S(V)t^2 + \frac{1}{3} \psi^3_S(V)t^3 + \cdots \\
= \log(1 + S^1(V)t + S^2(V)t^2 + \cdots),
\]

\[
\psi^1_\Lambda(V)t - \frac{1}{2} \psi^2_\Lambda(V)t^2 + \frac{1}{3} \psi^3_\Lambda(V)t^3 - \cdots \\
= \log(1 + \Lambda^1(V)t + \Lambda^2(V)t^2 + \cdots).
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Define \(\psi^m_S(V)\) and \(\psi^n_{\Lambda}(V)\) in \(\mathbb{Q} \otimes R_{KG}\) by

\[
\psi^1_S(V)t + \frac{1}{2}\psi^2_S(V)t^2 + \frac{1}{3}\psi^3_S(V)t^3 + \cdots \\
= \log(1 + S^1(V)t + S^2(V)t^2 + \cdots),
\]

\[
\psi^1_\Lambda(V)t - \frac{1}{2}\psi^2_\Lambda(V)t^2 + \frac{1}{3}\psi^3_\Lambda(V)t^3 - \cdots \\
= \log(1 + \Lambda^1(V)t + \Lambda^2(V)t^2 + \cdots).
\]

It turns out that \(\psi^n_S(V), \psi^n_{\Lambda}(V) \in R_{KG}\) and

\[
\psi^n_S(U + V) = \psi^n_S(U) + \psi^n_S(V), \quad \psi^n_{\Lambda}(U + V) = \psi^n_{\Lambda}(U) + \psi^n_{\Lambda}(V).
\]

Thus we get \(\mathbb{Z}\)-linear functions called the **Adams operations**:

\[
\psi^n_S, \psi^n_{\Lambda} : R_{KG} \to R_{KG}.
\]
Adams operations

Clearly $\psi^1_S(V), \ldots, \psi^n_S(V)$ are polynomials in $S^1(V), \ldots, S^n(V)$ and vice versa. Similarly for the exterior powers.

Thus knowledge of the symmetric and exterior powers in $R_{KG}$ is equivalent to knowledge of the Adams operations (assuming we know how to multiply in $R_{KG}$).
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**Problem.** For given $G$ and $K$ determine $\psi^n_S$ and $\psi^n_\Lambda$. 
Adams operations

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Thus knowledge of the symmetric and exterior powers in $R_{KG}$ is equivalent to knowledge of the Adams operations (assuming we know how to multiply in $R_{KG}$).

**Problem.** For given $G$ and $K$ determine $\psi^*_S$ and $\psi^*_\Lambda$.

Of course, this is a bit of a cheat! Our only definition of the Adams operations involves the symmetric and exterior powers.
Clearly \( \psi^1_S(V), \ldots, \psi^n_S(V) \) are polynomials in \( S^1(V), \ldots, S^n(V) \) and vice versa. Similarly for the exterior powers.

Thus knowledge of the symmetric and exterior powers in \( R_{KG} \) is equivalent to knowledge of the Adams operations (assuming we know how to multiply in \( R_{KG} \)).

**Problem.** For given \( G \) and \( K \) determine \( \psi^n_S \) and \( \psi^n_\Lambda \).

Of course, this is a bit of a cheat! Our only definition of the Adams operations involves the symmetric and exterior powers.

For now it is perhaps best to think of Adams operations as providing an attractive re-packaging of results on exterior and symmetric powers rather than a tool for proving theorems about these modules.
The main properties of the Adams operations on $R_{KG}$ were given by Benson (1984) and RMB (2003) following ideas of Adams, Frobenius and others.

**Linearity.**
As we have seen, $\psi^n_S$ and $\psi^n_\Lambda$ are $\mathbb{Z}$-linear maps.

‘Nice’ behaviour when $n$ is not divisible by $p$.
For $p \nmid n$, $\psi^n_S = \psi^n_\Lambda$, and $\psi^n_S$ is a ring endomorphism of $R_{KG}$.

**Factorisation property.**
If $n = kp^d$ where $p \nmid k$ then

$$\psi^n_S = \psi^k_S \circ \psi^{p^d}_S, \quad \psi^n_\Lambda = \psi^k_\Lambda \circ \psi^{p^d}_\Lambda.$$
Theorem 1. $\psi^n_\Lambda$ is periodic in $n$ if and only if the Sylow $p$-subgroups of $G$ are cyclic.

The proof is fairly elementary, relying on the facts that if the Sylow $p$-subgroups are cyclic then there are only finitely many indecomposables (Higman) and the Green ring is semi-simple (Green and O’Reilly).
Theorem 1. \( \psi_A^n \) is periodic in \( n \) if and only if the Sylow \( p \)-subgroups of \( G \) are cyclic.

The proof is fairly elementary, relying on the facts that if the Sylow \( p \)-subgroups are cyclic then there are only finitely many indecomposables (Higman) and the Green ring is semi-simple (Green and O’Reilly).

There is also a corresponding result for \( \psi_S^n \).

Theorem 2. \( \psi_S^n \) is periodic in \( n \) if and only if the Sylow \( p \)-subgroups of \( G \) are cyclic.

The proof of this is more difficult. It relies on deep work of Symonds (2007), based on previous work of Karagueuzian and Symonds.
Suppose now that the Sylow $p$-subgroups of $G$ are cyclic. Thus $\psi_S^n$ and $\psi_A^n$ are both periodic in $n$ and we would like to calculate the minimum periods. Let $e$ denote the exponent of $G$. 
Minimum period

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Minimum period

Suppose now that the Sylow $p$-subgroups of $G$ are cyclic. Thus $\psi_S^n$ and $\psi_\Lambda^n$ are both periodic in $n$ and we would like to calculate the minimum periods. Let $e$ denote the exponent of $G$.

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$G$ a cyclic $p$-group.

(i) $\psi_S^n$ is periodic in $n$ with minimum period lcm(2, $e$);

(ii) $\psi_\Lambda^n$ is periodic in $n$ with minimum period $2e$. 


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$G$ a cyclic $p$-group.

(i) $\psi^n_S$ is periodic in $n$ with minimum period $\text{lcm}(2, e)$;

(ii) $\psi^n_\Lambda$ is periodic in $n$ with minimum period $2e$.

$G$ has proper cyclic Sylow $p$-subgroup.

We obtain a lower bound; $\psi^n_S$ and $\psi^n_\Lambda$ are periodic in $n$ with minimum period divisible by $\text{lcm}(2, e)$. 

Let $G$ be a cyclic $p$-group of order $q > 1$. Recall that $R_{KG}$ has $\mathbb{Z}$-basis \{${V_1, V_2, \ldots, V_q}$\}.

What are $\psi^n_S(V_r)$ and $\psi^n_\Lambda(V_r)$?

We start with the case where $p \nmid n$ and write $\psi^n = \psi^n_S = \psi^n_\Lambda$.
Let $G$ be a cyclic $p$-group of order $q > 1$.
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We start with the case where $p \nmid n$ and write $\psi^n = \psi^n_S = \psi^n_\Lambda$.

**Theorem 3.** Suppose that $p \nmid n$ and let $r \in \{1, \ldots, q\}$.
Write $r = kp^i + s$ where $1 \leq k \leq p - 1$ and $1 \leq s \leq p^i$.
Then there is a formula (involving only elementary arithmetic) giving $\psi^n(V_r)$ in terms of $\psi^n(V_s)$ and $\psi^n(V_{p^i - s})$.

(Here we take $V_0 = 0$ to cover the case where $p^i - s = 0$.)
This theorem gives $\psi^n(V_r)$ recursively on $r$. 
Cyclic $p$-groups

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(Here we take $V_0 = 0$ to cover the case where $p^i - s = 0$.)
This theorem gives $\psi^n(V_r)$ recursively on $r$.

The proof uses and extends work of Almkvist & Fossum, Kouwenhoven, Hughes & Kemper, and Gow & Laffey.
Patterns for cyclic $p$-groups

When we calculated $\psi^n$ using Theorem 3 we noticed some interesting patterns, which we were later able to prove.

Example. Let $G = C_{25}$ where $p = 5$.

\[
\begin{align*}
\psi^3(V_1) &= V_1 \\
\psi^3(V_2) &= V_4 - V_2 \\
\psi^3(V_3) &= V_5 - V_3 + V_1 \\
\psi^3(V_4) &= V_4 \\
\psi^3(V_5) &= V_5 \\
\psi^3(V_6) &= V_{16} - V_{14} + V_4 \\
\psi^3(V_7) &= V_{19} - V_{17} + V_{13} - V_{11} + V_5 - V_3 + V_1 \\
\psi^3(V_8) &= V_{20} - V_{18} + V_6 - V_14 + V_{12} - V_{10} + V_4 - V_2 \\
\psi^3(V_9) &= V_{19} - V_{11} + V_1 \\
\psi^3(V_{10}) &= V_{20} - V_{10} \\
\psi^3(V_{11}) &= V_{21} - V_{11} + V_1 \\
\psi^3(V_{12}) &= V_{24} - V_{22} + V_{20} - V_{14} + V_{12} - V_{10} + V_4 - V_2 \\
\psi^3(V_{13}) &= V_{25} - V_{23} + V_21 - V_{15} + V_{13} - V_{11} + V_5 - V_3 + V_1
\end{align*}
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Patterns for cyclic $p$-groups

When we calculated $\psi^n$ using Theorem 4 we noticed some interesting patterns, which we were later able to prove.

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\end{align*}
\]
Recall that for a $KG$-module $V$, $\Omega(V)$ is defined up to isomorphism as the kernel of any map $P(V) \to V$ where $P(V)$ is the projective cover of $V$.

Hence

$$\Omega(V_r) = V_{q-r} \text{ for } r = 1, \ldots, q$$

with the convention that $V_0 = 0$. 
Heller translates

Recall that for a $KG$-module $V$, $\Omega(V)$ is defined up to isomorphism as the kernel of any map $P(V) \twoheadrightarrow V$ where $P(V)$ is the projective cover of $V$.
Hence

$$\Omega(V_r) = V_{q-r} \text{ for } r = 1, \ldots, q$$

with the convention that $V_0 = 0$.

We extend $\Omega$ to a $\mathbb{Z}$-linear map $\Omega : R_{KG} \rightarrow R_{KG}$. Also we write $\Omega^n$ for the composite of $\Omega$ taken $n$ times. It is easily seen that

$$\Omega^n(V) = \begin{cases} V + aV_q & \text{if } n \text{ is even}, \\ \Omega(V) + aV_q & \text{if } n \text{ is odd}, \end{cases}$$

where $a$ is some integer.
Reduction of $\psi^n_S$ to $\psi^n_\Lambda$

Peter Symonds (2007) gave a recursive way of finding $S^n(V_r)$ in terms of exterior powers. His result leads to a corresponding result for Adams operations which is somewhat easier to state.
Reduction of $\psi_S^n$ to $\psi_\Lambda^n$

Peter Symonds (2007) gave a recursive way of finding $S^n(V_r)$ in terms of exterior powers. His result leads to a corresponding result for Adams operations which is somewhat easier to state.

**Theorem 4.** Suppose that $q/p \leq r \leq q$. Then, for all $n$,

$$
\psi_S^n(V_r) = (-1)^{n-1} \Omega^n(\psi_\Lambda^n(V_{q-r})) + (n,q)V_q/(n,q) + cV_q
$$

where the integer $c$ may be calculated by a dimension count if $\psi_\Lambda^n(V_{q-r})$ is known and $(n,q)$ denotes the gcd of $n$ and $q$. 
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\psi^n_S(V_r) = (-1)^{n-1}\Omega^n(\psi^n_\Lambda(V_{q-r})) + (n,q)V_{q/(n,q)} + cV_q
$$

where the integer $c$ may be calculated by a dimension count if $\psi^n_\Lambda(V_{q-r})$ is known and $(n,q)$ denotes the gcd of $n$ and $q$.

This is easily seen to give $\psi^n_S$ in terms of $\psi^n_\Lambda$.
(For $r < q/p$ the module $V_r$ may be regarded as a module for a proper factor group of $G$.)

Reduction of $\psi^n_S$ to $\psi^n_\Lambda$
The Adams operations are certain linear maps on the Green ring $R_{KG}$ that encapsulate the behaviour of symmetric and exterior powers.
Recap

- The Adams operations are certain linear maps on the Green ring $R_{KG}$ that encapsulate the behaviour of symmetric and exterior powers.
- The Adams operations have many nice properties.

- We have shown that $\psi_n S$ and $\psi_n \Lambda$ are periodic in $n$ if and only if the Sylow $p$-subgroups of $G$ are cyclic.
- We gave a lower bound for the minimum periods.
- When $G$ is a cyclic $p$-group we gave recursive formula to calculate $\psi_n S = \psi_n \Lambda$ for $n$ not divisible by $p$.
- This recursion gives rise to some nice patterns.
- For cyclic $p$-groups we also showed that $\psi_n S (V_r)$ can be expressed in terms of $\psi_n \Lambda (V_q - r)$, where $V_q - r$ is the Heller translate of $V_r$. 
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Recap

- The Adams operations are certain linear maps on the Green ring $R_{KG}$ that encapsulate the behaviour of symmetric and exterior powers.
- The Adams operations have many nice properties.
- We have shown that $\psi^n_S$ and $\psi^n_\Lambda$ are periodic in $n$ if and only if the Sylow $p$-subgroups of $G$ are cyclic. We gave a lower bound for the minimum periods.
- When $G$ is a cyclic $p$-group we gave recursive formula to calculate $\psi^n_S = \psi^n_\Lambda$ for $n$ not divisible by $p$. This recursion gives rise to some nice patterns.
- For cyclic $p$-groups we also showed that $\psi^n_S(V_r)$ can be expressed in terms of $\psi^n_\Lambda(V_{q-r})$, where $V_{q-r}$ is the Heller translate of $V_r$. 
The determination of $\Lambda^n(V_r)$ and $\psi^n\Lambda(V_r)$ for a cyclic $p$-group is still open in general. Frank Himstedt and Peter Symonds have recently discovered a way of evaluating $\Lambda^n(V_r)$ in the case $p = 2$. This leads to a description of $\psi^n\Lambda$ as follows.

- It can be shown that $\psi^n\Lambda$ is equal to the identity function for all odd $n$.
- Also, if $n = k2^d$ where $k$ is odd then $\psi^n\Lambda = \psi^k\Lambda \circ \psi^{2^d}\Lambda$.
- Thus it remains to describe $\psi^{2^d}\Lambda$ for $d \geq 1$.

**Theorem 5.** Let $G$ be a cyclic 2-group. Write $r = 2^i + s$ where $1 \leq s \leq 2^i$. Then

$$\psi^2\Lambda(V_r) = 2V_{2i+1} - 2V_{2i+1-s} + \psi^2\Lambda(V_{2i-s})$$

and

$$\psi^{2^d}\Lambda(V_r) = 2\psi^{2^{d-1}}\Lambda(V_s) + \psi^{2^d}\Lambda(V_{2i-s}) \text{ for } d \geq 2.$$

($\psi^2\Lambda$ can also be obtained from work of Gow and Laffey (2006)).