Tropical Linear Algebra (Achievements and Challenges)

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III. Tropical permanent

Tropical permanent - basics

$A = (a_{ij}) \in \overline{\mathbb{R}}^n$
(Conventional) permanent:

$$per(A) = \sum_{\pi \in P_n} \prod_{i \in N} a_{i,\pi(i)}$$

Tropical permanent:

$$tper(A) = \sum\nolimits^{\oplus}_{\pi \in P_n} \prod\nolimits^{\otimes}_{i \in N} a_{i,\pi(i)}$$

where P_n ... the set of permutations of $N = \{1, ..., n\}$
In conventional notation:

$$tper(A) = \max_{\pi \in P_n} \sum_{i \in N} a_{i,\pi(i)}$$

The weight of $\pi \in P_n$:

$$w(\pi, A) = \prod\nolimits^{\otimes}_{i \in N} a_{i,\pi(i)} = \sum_{i \in N} a_{i,\pi(i)}$$

Tropical permanent - basics

$\pi \in P_n$ is *optimal* if

$$w(\pi, A) = tper(A)$$

The task of finding an optimal permutation ... *assignment problem* ... *Hungarian method* $O\left(n^3\right)$
Aspects and variants in: *Burkard RE, Dell'Amico M, Martello S (2009) Assignment problems. SIAM, Philadelphia*

Tropical permanent - basics

Denote

$$ap(A) = \{\pi \in P_n; w(\pi, A) = tper(A)\}$$

Lemma: If $A, C, D \in \overline{\mathbb{R}}^{n \times n}$ and C and D are diagonal matrices then
$ap(C \otimes A \otimes D) = ap(A)$ and
$tper(C \otimes A \otimes D) = tper(C) \otimes tper(A) \otimes tper(D)$

Tropical permanent - basics

Theorem (Hungarian Method) For every $A \in \overline{\mathbb{R}}^{n \times n}$ with $tper(A) > \varepsilon$ there exist diagonal matrices C and D such that

$B \overset{df}{=} C \otimes A \otimes D \leq 0$

$tper(C \otimes A \otimes D) = 0$ and (hence)

$ap(A) = \left\{ \pi \in P_n; b_{i,\pi(i)} = 0 \text{ for all } i \in N \right\}$

C and D can be found in $O\left(n^3\right)$ time

Link to nonnegative matrices

Dequantisation: $\left(a^k + b^k\right)^{1/k} \longrightarrow \max\left(a, b\right)$ for $k \longrightarrow \infty$
A ... a nonnegative matrix
$\left\{A^k\right\}_{k=1}^{\infty}$ ... sequence of *Hadamard (Schur)* powers

$$\left(per\left(A^k\right)\right)^{1/k} \longrightarrow \max_{\pi \in P_n} \prod_{i \in N} a_{i,\pi(i)} = \sum_{\pi \in P_n}{}^{\oplus} \prod_{i \in N}{}^{\otimes} a_{i,\pi(i)} = tper\left(A\right)$$

in max-times

Van der Waerden Conjecture

Among all doubly stochastic $n \times n$ matrices the permanent obtains its minimum for the matrix $A = (a_{ij})$, where $a_{ij} = \frac{1}{n}$ for all $i, j \in N$.

Among all doubly stochastic $n \times n$ matrices the **tropical** permanent obtains its minimum for the matrix $A = (a_{ij})$, where $a_{ij} = \frac{1}{n}$ for all $i, j \in N$.

$v_1, \dots, v_n \in \overline{\mathbb{R}}^m$ are called

(Weakly) *LD* iff for some k and $\alpha_j \in \overline{\mathbb{R}}$

$$v_k = \sum\nolimits_{j \neq k}^{\oplus} \alpha_j \otimes v_j$$

Gondran-Minoux LD iff for some
$U, V \subseteq \{1, \dots, n\}$, $U \cap V = \emptyset$, $U, V \neq \emptyset$ and $\alpha_j \in \overline{\mathbb{R}}$

$$\sum\nolimits_{j \in U}^{\oplus} \alpha_j \otimes v_j = \sum\nolimits_{j \in V}^{\oplus} \alpha_j \otimes v_j$$

Strongly LD iff

$$\sum\nolimits_{j=1,\dots,n}^{\oplus} v_j \otimes x_j = v$$

does not have a unique solution for any $v \in \mathbb{R}^m$

$\omega \in \{\text{strong, GM, weak}\}$

$\omega-$independent $\equiv$ not $\omega-$dependent

SLI $\implies$ GMLI $\implies$ WLI

Relations between independences

SLI $\Longrightarrow$ GMLI $\Longrightarrow$ WLI

SLI $\Longrightarrow$ GMLI $\Longrightarrow$ $m \geq n$

WLI does not imply $m \geq n$ in general:

$v_1, \ldots, v_n \in \overline{\mathbb{R}}^m$ may be WLI for any n if $m \geq 3$ ("dimensional anomaly")

WLI $\implies$ $m \geq n$ if $m = 2$

Checking independence

No polynomial method for checking SLI or GMLI is known (?)

OPEN: Are SLI and GMLI polynomially verifiable or *NP*-complete?

LI can be checked in $O\left(mn^2\right)$ time

Types of regularity

$\omega-$*regular* matrix ... square matrix with $\omega-$independent columns

A is strongly regular $\Longrightarrow A$ is GM regular $\Longrightarrow$ regular

$\omega-$rank of an $m \times n$ matrix A is the greatest r such that A contains an $r \times r$ $\omega-$regular submatrix

Independence of columns and full column rank for an $m \times n$ matrix A :

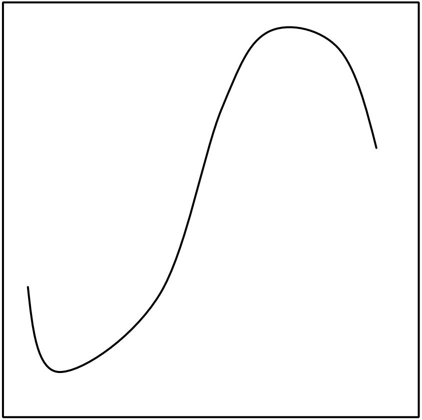
A has SLI columns $\Longleftrightarrow A$ has full strong rank

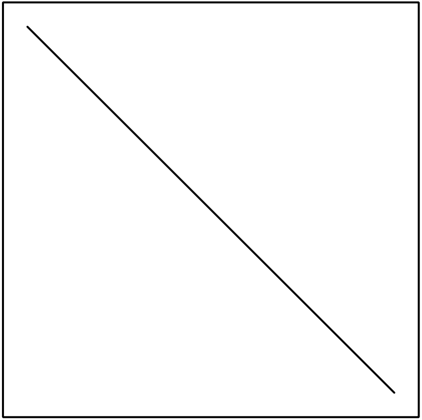
A has GMLI columns $\Longleftarrow$ A has full GM rank

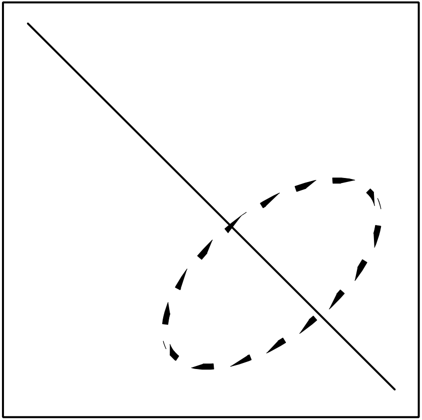
$\Longrightarrow$ disproved by S.Gaubert (2003)

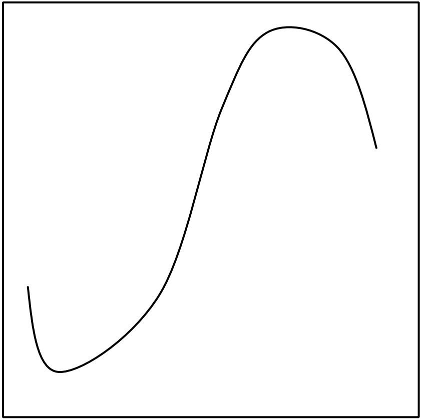
Criteria for checking regularity

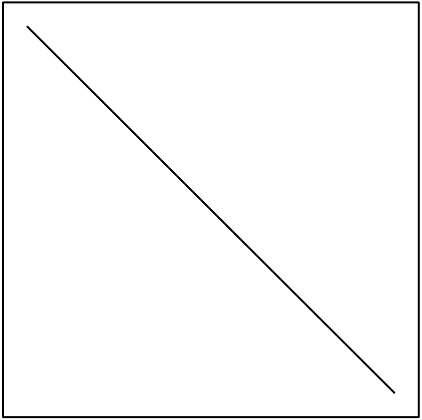
A is strongly regular $\iff |ap(A)| = 1$
A is GM regular $\iff$ all permutations in $ap(A)$ are of the same parity

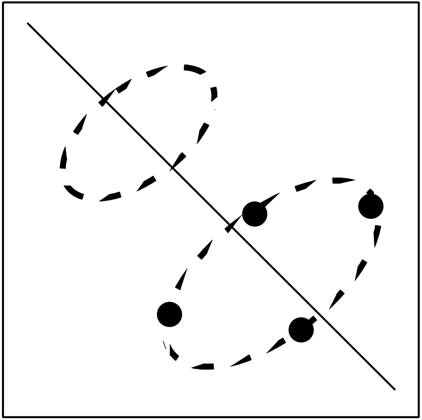












Criteria for checking regularity

A strongly regular $\Longleftrightarrow$
$|ap(A)| = 1 \iff$
D_A does not contain a *cycle*
A GM regular $\Longleftrightarrow$
$ap(A) \subseteq P_n^+$ or $ap(A) \subseteq P_n^- \iff$
D_A does not contain an *even cycle*

Criteria for checking regularity

Sign of a cyclic permutation (cycle) $\sigma = (i_1 i_2 ... i_k)$ ...
$sgn(\sigma) = (-1)^{k-1}$
σ has odd parity $\iff$ sign of σ is -1 $\iff$ k even
If $\pi = \pi_1 \circ ... \circ \pi_r$ then the *sign* of π is

$$sgn(\pi) = sgn(\pi_1) . \; ... \; .sgn(\pi_r)$$

$sgn(\pi) = -1$ $(+1)$ ... π is *odd* (*even*)
π odd $\Longrightarrow$ at least one of the constituent cycles is even

Criteria for checking regularity

The set of odd (even) permutations ... P_n^- (P_n^+)

$$
\begin{aligned}
ap^+(A) &= ap(A) \cap P_n^+ \\
ap^-(A) &= ap(A) \cap P_n^-
\end{aligned}
$$

$$
\begin{aligned}
tper^+(A) &= \max_{\pi \in P_n^+} \sum_{i \in N} a_{i,\pi(i)} \\
tper^-(A) &= \max_{\pi \in P_n^-} \sum_{i \in N} a_{i,\pi(i)}
\end{aligned}
$$

Criteria for checking regularity

A is Gondran-Minoux regular $\iff$
$ap^{+}(A) = \emptyset$ or $ap^{-}(A) = \emptyset \iff$
$tper^{+}(A) \neq tper^{-}(A) \iff$
D_A does *not* contain an even cycle ... can be checked in $O\left(n^3\right)$ time

Parity assignment problem

OPEN: Find **both** $tper^{+}(A)$ and $tper^{-}(A)$
Note: $tper^{+}(A) \oplus tper^{-}(A) = tper(A)$
Solvable in linear time (after solving the assignment problem) for
Monge matrices
diagonally dominant symmetric matrices

Tropical polynomials

Tropical polynomials:

$$p(z) = a_0 \oplus a_1 \otimes z \oplus \ldots \oplus a_n \otimes z^n$$

But also

$$8.3 \otimes z^{-7.2} \oplus (-2.6) \otimes z^{3.7} \oplus 6.5 \otimes z^{12.3}$$

In general, a tropical polynomial is

$$p(z) = \sum\nolimits_{r=0,\ldots,p}^{\oplus} c_r \otimes z^{j_r}, \;\; c_r, j_r \in \mathbb{R}$$

j_p ... the *degree* and
$p + 1$ ... the *length* of $p(z)$

Tropical polynomials

$p(z) = \sum_{r=0,\ldots,p}^{\oplus} c_r \otimes z^{j_r}$
$p(z)$ ... formal algebraic expressions with z as an indeterminate
or
$p(z)$ ... tropical functions of z
In conventional notation $p(z)$:

$$\max_{r=0,\ldots,p} (c_r + j_r z)$$

Theorem (Cuninghame-Green, 1983) Every tropical polynomial can be factorised to linear terms in linear time in the degree p.

Inessential terms

$(a \oplus b)^k = a^k \oplus b^k$
$(1 \oplus z)^2 = 2 \oplus z^2$
$(1 \oplus z)^2 = 2 \oplus 1 \otimes z \oplus z^2$
$2 \oplus z^2 = 2 \oplus 1 \otimes z \oplus z^2$ as functions
$1 \otimes z \leq 2 \oplus z^2$
The term $c_s \otimes z^{j_s}$ of a tropical polynomial $\sum^{\oplus}_{r=0,\ldots,p} c_r \otimes z^{j_r}$ is called *inessential* if

$$c_s \otimes z^{j_s} \leq \sum_{r \neq s}^{\oplus} c_r \otimes z^{j_r}$$

holds for every $z \in \mathbb{R}$ and *essential* otherwise
Inessential terms can be omitted if tropical polynomials are considered as functions

Tropical characteristic polynomial

Tropical characteristic polynomial for $A \in \overline{\mathbb{R}}^{n \times n}$:

$$\begin{aligned}
\chi_A(\lambda) &= tper(A \oplus \lambda \otimes I) \\
&= tper \begin{pmatrix} a_{11} \oplus \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} \oplus \lambda & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \oplus \lambda \end{pmatrix} \\
&= \delta_n \oplus \delta_{n-1} \otimes \lambda \oplus \dots \oplus \delta_1 \otimes \lambda^{n-1} \oplus \lambda^n
\end{aligned}$$

$\chi_A(\lambda) = (\lambda \oplus c_1) \otimes (\lambda \oplus c_2) \otimes \dots \otimes (\lambda \oplus c_n)$
$c_1, \dots, c_n$ ... corners of $\chi_A(\lambda)$
All eigenvalues are corners (but not all corners are eigenvalues!)
$\max_i c_i = \lambda(A)$

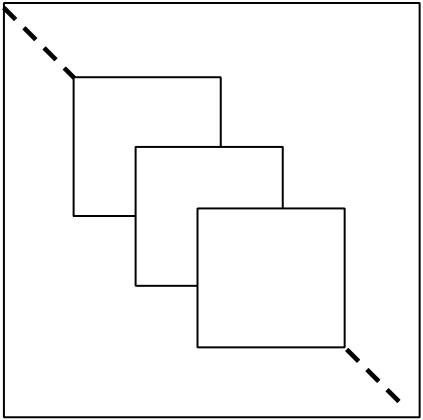
Calculating the coefficients of a tropical characteristic polynomial

If $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$ then for $k = 1, ..., n$:

$$\delta_k = \max \{tper(B); B \text{ is a } k \times k \textbf{ principal} \text{ submatrix of } A\}$$

$\delta_n = tper(A)$
$\delta_1 = \max(a_{11}, a_{22}, ..., a_{nn})$
$\chi_A(\lambda) = \lambda^n$ if and only if D_A is acyclic

Calculating the coefficients of a tropical characteristic polynomial



Calculating the coefficients of a tropical characteristic polynomial

Checking all $\binom{n}{k}$ principal submatrices $\longrightarrow$ polynomial for a fixed k

$$\sum_{k=1}^{n} \binom{n}{k} = 2^n - 1$$

No polynomial method for finding δ_k for all k (?)

Modification obtained after removing the word "principal" is $O\left(n^3\right)$ for every k

Calculating the coefficients of a tropical characteristic polynomial

All **essential terms** of a tropical characteristic polynomial can be found in $< O(n^4)$ time

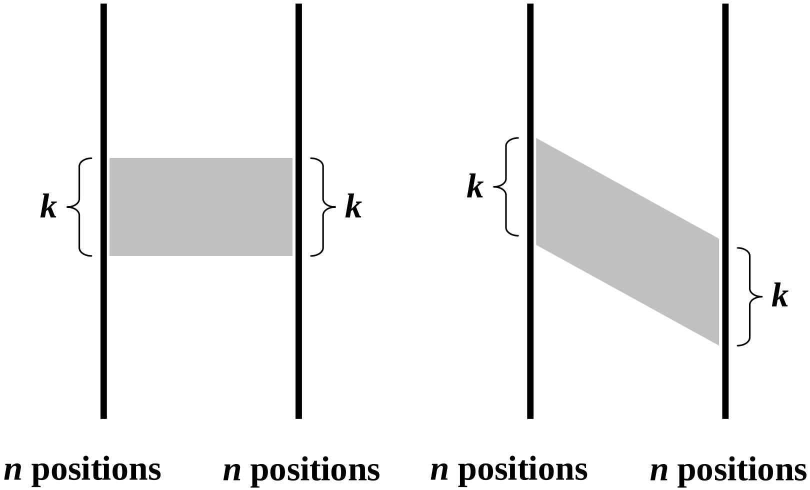
OPEN: Can all coefficients be found in polynomial time?

Related combinatorial problem:

Given a digraph D with n nodes and a positive integer k $(k < n)$, is there a set of pairwise node-disjoint cycles covering exactly k nodes of D?

A company with n employees requires workers to swap their jobs
To avoid disruption: only k $(k < n)$, should actually swap their jobs
a_{ij} ... *costs* associated with the swaps $\longrightarrow$ min
a_{ij} ... *measures of preference* of the swaps $\longrightarrow$ max
The *best principal submatrix problem* (BPSM):
Given a real $n \times n$ *matrix* A *and a* $k < n$, *find a* $k \times k$ *principal submatrix of* A *whose optimal assignment problem value is maximal.*

An application: the job rotation problem



$A = (a_{ij}) \in \mathbb{R}^{n \times n}$, $v \in \mathbb{R}$

$$
\begin{aligned}
p^{+}(A, v) &= \left|\{\pi \in P_n^{+}; w(\pi, A) = v\}\right| \\
p^{-}(A, v) &= \left|\{\pi \in P_n^{-}; w(\pi, A) = v\}\right|
\end{aligned}
$$

$$P_k(A) = \{B; B \text{ is a } k \times k \text{ principal submatrix of } A\}$$

Tropical Cayley-Hamilton

The *tropical characteristic equation* for A:

$$\lambda^n \oplus \sum\nolimits_{k \in J}^{\oplus} c_{n-k} \otimes \lambda^k = c_1 \otimes \lambda^{n-1} \oplus \sum\nolimits_{k \in \overline{J}}^{\oplus} c_{n-k} \otimes \lambda^k$$

where

$$c_k = \max \left\{ v; \sum_{B \in P_k(A)} p^+(B, v) \neq \sum_{B \in P_k(A)} p^-(B, v) \right\}, \ k = 1, ..., n$$

$$J = \{j; d_j > 0\}, \ \overline{J} = \{j; d_j < 0\}$$

$$d_k = (-1)^k \left(\sum_{B \in P_k(A)} p^+(B, c_k) - \sum_{B \in P_k(A)} p^-(B, c_k) \right), k = 1, ..., n$$

Tropical Cayley-Hamilton

Theorem (Tropical Cayley-Hamilton, Olsder and Roos, 1988)
Every real square matrix A satisfies its tropical characteristic equation.

Finding the tropical characteristic equation is usually hard
A special case: If $ap(A) = \{id\}$ then the tropical characteristic equation of A is:

$$\begin{aligned} & \lambda^n \oplus \delta_2 \otimes \lambda^{n-2} \oplus \delta_4 \otimes \lambda^{n-4} \oplus ... \\ = \quad & \delta_1 \otimes \lambda^{n-1} \oplus \delta_3 \otimes \lambda^{n-3} \oplus \delta_5 \otimes \lambda^{n-5} \oplus ... \end{aligned}$$

where $\delta_k =$ the sum of the k greatest diagonal values of A

An escape from max-plus: max-min ("bottleneck algebra")

$a \oplus b = \max(a, b)$
$a \otimes b = \min(a, b)$
$a, b \in \mathbb{B}$
$(\mathbb{B}, \leq)$ ... dense linearly ordered set without maximum and minimum

Bottleneck assignment problem (bottleneck tropical permanent)

Assignment problem $\longrightarrow$ *Bottleneck* assignment problem
Bottleneck permanent:

$$bper(A) = \sum\nolimits_{\pi \in P_n}^{\oplus} \prod\nolimits_{i \in N}^{\otimes} a_{i,\pi(i)} = \max_{\pi \in P_n} \min_{i \in N} a_{i,\pi(i)}$$

$w(\pi, A) = \min_{i \in N} a_{i,\pi(i)}$
$bap(A) = \{\pi \in P_n; w(\pi, A) = bper(A)\}$
Best computational complexity: $O\left(n^2\sqrt{n/\log n}\right)$

Trapezoidal matrices

$A \oplus B = \ldots \quad A \otimes B = \ldots$

$A \in \mathbb{B}^{m\times n}$ has SLI columns if $A \otimes x = b$ has a unique solution for at least one $b \in \mathbb{B}^{m}$

A is strongly regular if it has SLI columns and $m = n$

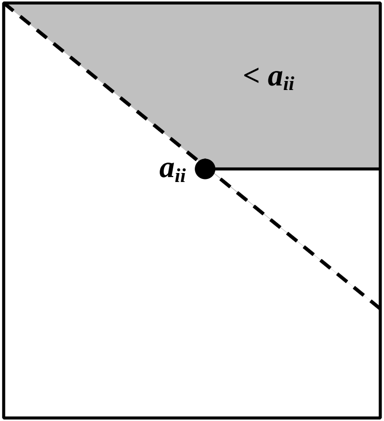
A is said to be similar to B ($A \sim B$) if A can be obtained from B by permuting the rows and/or columns (independently)

$A = (a_{ij})$ is called *trapezoidal* if

$$a_{ii} > a_{kj} \text{ for every } i, j, k, \;\; k \leq i \text{ and } k < j$$

Trapezoidal matrices

$a_{ii} > a_{kj}$ for every $i, j, k,\ \ k \leq i$ and $k < j$



Trapezoidal matrices

Theorem (PB et al, 1987) Let $A \in \mathbb{B}^{m \times n}$. Then
A has SLI columns $\iff A \sim B$, B trapezoidal
$m = n$, A has strong permanent $\implies A \sim B$, B trapezoidal
Corollary: $m = n$, A has strong permanent $\implies A$ is strongly regular (but not $\impliedby$)
Strong bottleneck permanent can be checked in $O\left(n^2 \log n\right)$ time
Corollary: Bottleneck assignment problem can be solved in $O\left(n^2 \log n\right)$ time if there is exactly one optimal permutation.

Thank you!