Tropical Linear Algebra (Achievements and Challenges)

Peter Butkovic

I. **Basics**
II. **Reachability of tropical eigenspaces by matrix orbits**
III. **Tropical permanent**

Credits

R.A.Cuninghame-Green
N.N.Vorobyev
M.Gondran
M.Minoux
G.Cohen
J.-P.Quadrat
V.Maslov
V.Kolokoltsov
E.Wagneur
S.Gaubert
M.Akian
...

Part I - Tropical linear algebra basics

Max-plus and variants

$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$

$a \oplus b = \max(a, b)$

$a \otimes b = a + b$

$(\overline{\mathbb{R}}, \oplus, \otimes)$ ... idempotent, commutative semiring

Notation:

ε for $-\infty$

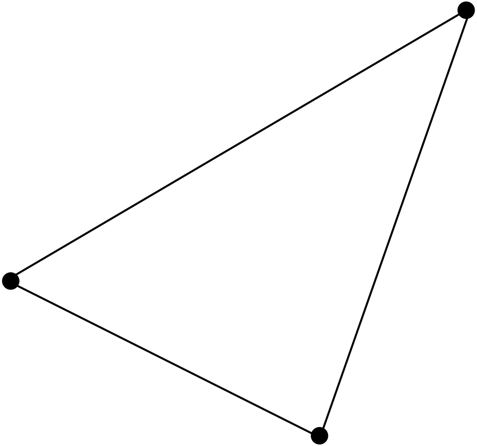
a^{-1} stands for $-a$

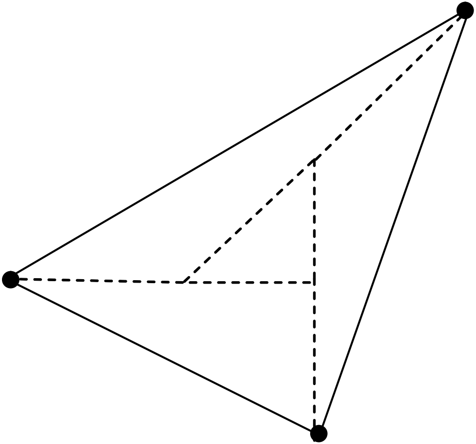
$$\underbrace{a \otimes a \otimes a \otimes \ldots \otimes a}_{k-\text{times}} = a^k$$

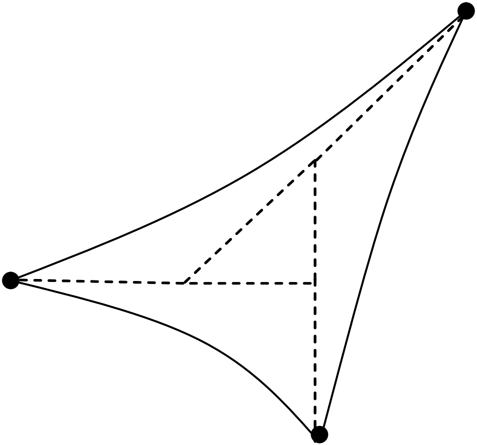
From classical to tropical...

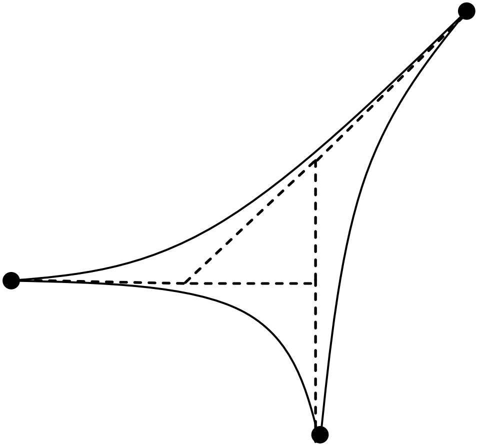
V.Maslov+V.Kolokoltsov (1980s): "dequantisation":

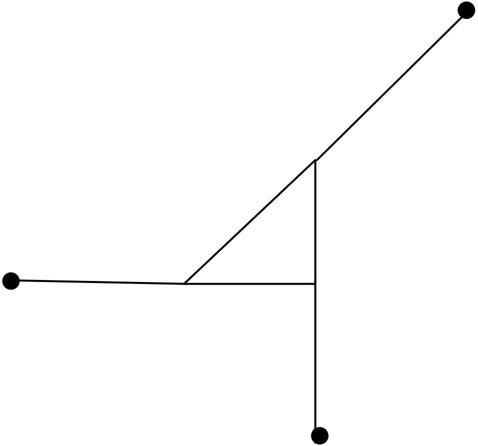
$$\left(a^k + b^k\right)^{1/k} \longrightarrow \max\left(a, b\right) \text{ for } k \longrightarrow \infty$$











Max-plus and variants

$\mathcal{G} = (G, \otimes, \leq)$ ... linearly ordered commutative group
$a \oplus b = \max(a, b)$
$\varepsilon \leq a$ for all $a \in G$ (adjoined)
$(G \cup \{\varepsilon\}, \oplus, \otimes)$ ... commutative idempotent semiring
$\mathcal{G}_0 = (\mathbb{R}, +, \leq)$ ... max-plus
$\mathcal{G}_1 = (\mathbb{R}, +, \geq)$ ... min-plus $(x \longrightarrow -x)$
$\mathcal{G}_2 = (\mathbb{R}^+, \cdot, \leq)$ ... max-times $(x \longrightarrow e^x)$
$\mathcal{G}_3 = (\mathbb{Z}, +, \leq)$
...
In what follows: $\mathcal{G}_0$

Extension to matrices and vectors

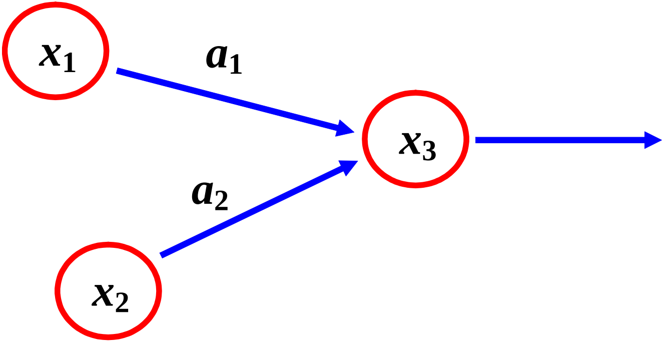
$$A \oplus B = (a_{ij} \oplus b_{ij})$$
$$A \otimes B = \left(\sum_k^{\oplus} a_{ik} \otimes b_{kj}\right)$$
$$\alpha \otimes A = (\alpha \otimes a_{ij})$$
$$diag(d_1, ..., d_n) = \begin{pmatrix} d_1 & & & & \\ & \ddots & & \varepsilon & \\ & & \ddots & & \\ & \varepsilon & & \ddots & \\ & & & & d_n \end{pmatrix}$$
$$I = diag(0, ..., 0)$$
$$\underbrace{A \otimes A \otimes A \otimes ... \otimes A}_{k-\text{times}} = A^k$$

Some basic properties

Compared to $(\mathbb{R}, +, .)$ we are
losing invertibility
gaining idempotency
A^{-1} exists $\iff$ A is a generalised permutation matrix
Idempotency: $a \oplus a = a$

$(a \oplus b)^k = a^k \oplus b^k$, if $k \geq 0$
$(A \oplus B)^k \neq A^k \oplus B^k$
$(I \oplus A)^k = I \oplus A \oplus A^2 \oplus ... \oplus A^k$
Another useful property: $A \leq B \Rightarrow A \otimes C \leq B \otimes C$

Tropical linear algebra: non-linear becomes "linear"



$$
\begin{aligned}
x_3 &= \max\left(x_1 + a_1, x_2 + a_2\right) \\
&= a_1 \otimes x_1 \oplus a_2 \otimes x_2 = \left(a_1, a_2\right) \otimes \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\end{aligned}
$$

Basic problems

One-sided max-linear systems:

$A \otimes x = b$

$A \otimes x \leq b$

$A \otimes x = \lambda \otimes x$ (x ... *eigenvector* if $x \neq \varepsilon$)

$A \otimes x \leq \lambda \otimes x$ (x ... *subeigenvector* if $x \neq \varepsilon$)

Basic problems

Two-sided max-linear systems:

$A \otimes x = B \otimes x$

$A \otimes x = B \otimes y$

$A \otimes x \oplus c = B \otimes x \oplus d$

$A \otimes x = \lambda \otimes B \otimes x$ (*generalized eigenproblem*)

Basic problems

Max-linear programming:

$f^T \otimes x \longrightarrow \min \ (\max)$

s.t.

$A \otimes x = b$

$f^T \otimes x \longrightarrow \min \ (\max)$

s.t.

$A \otimes x \oplus c = B \otimes x \oplus d$

Basic problems

Periodicity of matrix powers:
$A, A^2, A^3, \ldots$
Periodicity of matrix orbits:
$A \otimes x, A^2 \otimes x, A^3 \otimes x, \ldots$
Tropical polynomials, characteristic polynomial and Cayley-Hamilton
Linear independence, regularity, rank,...

Tools for working with tropical matrices

The conjugate and dual operators

Maximum cycle mean

Transitive closures

Permanent (tropical)

Dual operators and conjugation

Dual operators:

$$\begin{aligned} a \oplus' b &= \min(a, b) \\ a \otimes' b &= a + b \\ -\infty \otimes' +\infty &= +\infty = +\infty \otimes' -\infty \end{aligned}$$

The conjugate:

$$A^{\#} = -A^{T}$$

Theorem (Cuninghame-Green, 1979)

$$A \otimes x \leq b \Longleftrightarrow x \leq A^{\#} \otimes' b$$

Residuation, Galois connection, ...

Dual operators and conjugation

$$A \otimes x \leq b \Longleftrightarrow x \leq A^{\#} \otimes' b$$

Corollary 1: For any $A \in \overline{\mathbb{R}}^{m \times n}$ and $b \in \overline{\mathbb{R}}^{m}$ the system $A \otimes x \leq b$ has a solution and $\bar{x} \overset{df}{=} A^{\#} \otimes' b$ is the greatest solution.
Corollary 2: For any $A, B \in \overline{\mathbb{R}}^{m \times n}$

$$A \otimes \left(A^{\#} \otimes' B \right) \leq B$$

and [thus also]

$$A \otimes \left(A^{\#} \otimes' A \right) \leq A$$

Dual operators and conjugation

Remark: For every A actually

$$A \otimes \left(A^{\#} \otimes' A\right) = A$$

and more generally:

$$A^{\#} A A^{\#} A ... A^{\#} A A^{\#}$$
$$A A^{\#} A A^{\#} ... A A^{\#} A$$

$$\otimes \otimes' \otimes ... \otimes \otimes'$$
$$\otimes' \otimes \otimes' ... \otimes' \otimes$$

$$\left(A^{\#} \otimes A\right) \otimes' ... \otimes' \left(\left(A^{\#} \otimes A\right) \otimes' A\right) \otimes ... \otimes' A^{\#} = A^{\#}$$

Dual operators and conjugation

$\overline{x} = A^{\#} \otimes' b$ ... *the principal solution* to $A \otimes x \leq b$

What about $A \otimes x = b$?

Suppose $A \otimes x = b$ for some x

$\therefore A \otimes x \leq b$

$\therefore x \leq \overline{x}$

$\therefore A \otimes x \leq A \otimes \overline{x}$

$\therefore b = A \otimes x \leq A \otimes \overline{x} \leq b$

$\therefore A \otimes \overline{x} = b$

Corollary 3: $A \otimes x = b$ has a solution if and only if $A \otimes \overline{x} = b$ that is

$$A \otimes \left(A^{\#} \otimes' b \right) = b$$

$\overline{x} = A^{\#} \otimes' b$
For $j = 1, ..., n$

$$\overline{x}_j = \min_i \left(a_{ji}^{\#} + b_i \right)$$

$$= \min_i \left(-a_{ij} + b_i \right)$$

$$= - \max_i \left(a_{ij} - b_i \right)$$

$M_j = \left\{ k; \overline{x}_j = -a_{kj} + b_k \right\}, j = 1, ..., n$

Dual operators and conjugation

Combinatorial method (Cuninghame-Green, 1960): $A \otimes x = b$ if and only if $x \leq \overline{x}$ and

$$\bigcup_{x_j = \overline{x}_j} M_j = \{1, ..., m\}$$

Corollary: Finding a solution to $A \otimes x = b$ with the least number of components equal to $\overline{x} = A^{\#} \otimes' b$ is an NP-complete problem.

Maximum cycle mean

Given $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$, the *mean of a cycle* $\sigma = (i_1, ..., i_k, i_1)$:

$$\mu(\sigma, A) = \frac{a_{i_1 i_2} + a_{i_2 i_3} + ... + a_{i_k i_1}}{k}$$

Maximum cycle mean of $A \in \overline{\mathbb{R}}^{n \times n}$:

$$\lambda(A) = \max \{\mu(\sigma, A); \sigma \text{ cycle}\}$$

$\mu(\sigma, A) = \lambda(A)$ ... σ is *critical*

Many algorithms for the computation of $\lambda(A)$ (Karp's is $O(n^3)$)

Maximum cycle mean is the principal eigenvalue

For any A, $\lambda(A)$ is

an eigenvalue of A

the greatest (*principal*) eigenvalue of A

the only eigenvalue of A whose corresponding eigenvectors **may be finite**

the unique eigenvalue if A is irreducible (in this case all eigenvectors **are finite**)

Every eigenvalue of A is the maximum cycle mean of some principal submatrix

Maximum cycle mean - definite matrices

A is (*max*-)*definite* if $\lambda(A) = 0$

$\lambda(\alpha \otimes A) = \alpha \otimes \lambda(A)$

In particular: $\lambda\left((\lambda(A))^{-1} \otimes A\right) = (\lambda(A))^{-1} \otimes \lambda(A) = 0$

$A \longrightarrow A_\lambda = (\lambda(A))^{-1} \otimes A$ (transition to a definite matrix)

"Passage Theorem" (Friedland 1986)

A ... an irreducible nonnegative matrix
$\rho(A)$ ... the Perron root of A
$\{A^k\}_{k=1}^{\infty}$ ... sequence of *Hadamard (Schur)* powers
Dequantisation: $\left(\rho\left(A^k\right)\right)^{1/k} \longrightarrow \lambda(A)$ (in max-times) and

$$\lambda(A) \leq \rho(A) \leq n\lambda(A)$$

Associated graph

$$A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n} \longrightarrow D_A = (N, E, (a_{ij}))$$

where $E = \{(i,j); a_{ij} > -\infty\}$
A is *irreducible* iff D_A strongly connected

Critical graph

$\mu(\sigma, A) = \lambda(A)$ ... σ is *critical*
$C_A = (N, E_c)$ where E_c is the set of arcs of all critical cycles
N_c ... the set of nodes of critical cycles
$i \sim j$ (*equivalent* nodes) ... i and j belong to the same critical cycle

Transitive closures

For $A \in \overline{\mathbb{R}}^{n\times n}$ define:

$A^+ = \quad A \oplus A^2 \oplus A^3 \oplus \ldots$ (*metric matrix/weak transitive closure*)

$A^* = I \oplus A \oplus A^2 \oplus A^3 \oplus \ldots$ (*Kleene star/strong transitive closure*)

If A is definite:

$A^+ = \quad A \oplus A^2 \oplus \ldots \oplus A^{n-1} \oplus A^n$

$A^* = I \oplus A \oplus A^2 \oplus \ldots \oplus A^{n-1}$

Eigenspaces and subeigenspaces

$V(A,\lambda) = \{x \in \overline{\mathbb{R}}^n; A \otimes x = \lambda \otimes x\}, \lambda \in \overline{\mathbb{R}}$
$V^*(A,\lambda) = \{x \in \overline{\mathbb{R}}^n; A \otimes x \leq \lambda \otimes x\}, \lambda \in \overline{\mathbb{R}}$
$V(A)$ ... the set of all eigenvectors
$\Lambda(A)$ ... the set of all eigenvalues
Tropical subspace is $V \subseteq \overline{\mathbb{R}}^n$ if for $x, y \in V$ and $\alpha \in \overline{\mathbb{R}}$:
$x \oplus y \in V$ and
$\alpha \otimes x \in V$
$V(A,\lambda)$ and $V^*(A,\lambda)$ are (tropical) subspaces
Bases? Dimension?

Finitely generated tropical subspaces - bases and dimension

If $v_1, \ldots, v_k \in \overline{\mathbb{R}}^n, \alpha_1, \ldots, \alpha_k \in \overline{\mathbb{R}}$ then $\sum_{j=1,\ldots,k}^{\oplus} \alpha_j \otimes v_j$ is a *max-combination* of $v_1, \ldots, v_k$

For $M \in \overline{\mathbb{R}}^{m \times n}$ we denote $span\,(M) \overset{def}{=} \left\{ M \otimes z; z \in \overline{\mathbb{R}}^n \right\}$

$span\,(M)$ is a *(finitely generated)* subspace

Columns of M are called *generators of* $span\,(M)$

A *basis* of a finitely generated subspace is any set of generators such that none of them is a max-combination of the others

Dimension of a finitely generated subspace is the size of (any of) its basis

Eigenspaces and subeigenspaces - bases and dimension

Assume $A \in \overline{\mathbb{R}}^{n \times n}, \lambda(A) > \varepsilon$ and recall $A_\lambda = (\lambda(A))^{-1} \otimes A$
n_c ... number of critical nodes, that is $|N_c|$
n_{cc} ... number of **non-trivial** components of C_A
$n_0 = n - n_c$
Denote $(A_\lambda)^+$ by A_λ^+, $(A_\lambda)^*$ by A_λ^*
$\left(A_\lambda^+\right)_c =$ submatrix formed by the columns with critical indices
Note: A_λ^* is just $I \oplus A_\lambda^+$
Theorem:
$V(A, \lambda(A)) = span\left(\left(A_\lambda^+\right)_c\right)$ and $dim V(A, \lambda(A)) = n_{cc}$
$V^*(A, \lambda(A)) = span\left(A_\lambda^*\right)$ and $dim V(A, \lambda(A)) = n_{cc} + n_0$
Essentially unique bases of $V(A, \lambda(A))$ and $V^*(A, \lambda(A))$ can be found in $O\left(n^3\right)$ time

Finite subeigenvectors

Finite subeigenvectors may be important for applications:

$$V^{**}(A,\lambda)=\{x\in\mathbb{R}^n; A\otimes x\leq\lambda\otimes x\}, \lambda\in\overline{\mathbb{R}}$$

Theorem: Let $A=(a_{ij})\in\overline{\mathbb{R}}^{n\times n}, A\neq\varepsilon, \lambda\in\overline{\mathbb{R}}$. Then $V^{**}(A,\lambda)\neq\emptyset$ if and only if $\lambda\geq\lambda(A)$ and $\lambda>\varepsilon$.
If $\lambda\geq\lambda(A)$ and $\lambda>\varepsilon$ then

$$V^{**}(A,\lambda)=\left\{(\lambda^{-1}\otimes A)^*\otimes u; u\in\mathbb{R}^n\right\}.$$

An application: Bounded mixed-integer solution to a system of dual inequalities

BMISDI: *Find, or prove that it does not exist, a vector* $x = (x_1, ..., x_n)^T$ *satisfying:*

$$\left.\begin{array}{lr} x_i - x_j \geq b_{ij}, & (i, j \in N) \\ u_j \geq x_j \geq l_j, & (j \in N) \\ x_j \text{ integer}, & (j \in J) \end{array}\right\}$$

where $u = (u_1, ..., u_n)^T, l = (l_1, ..., l_n)^T \in \mathbb{R}^n$ *and* $J \subseteq N = \{1, ..., n\}$ *are given.*

An application: Bounded mixed-integer solution to a system of dual inequalities

The system of dual inequalities (SDI)

$$x_i - x_j \geq b_{ij} \quad (i, j \in N)$$

is equivalent to:

$$\max_{j \in N} (b_{ij} + x_j) \leq x_i \quad (i \in N)$$

in tropical notation:

$$\sum\nolimits_{j \in N}^{\oplus} b_{ij} \otimes x_j \leq x_i \ \ (i \in N)$$

or in the compact form:

$$B \otimes x \leq x$$

An application: Bounded mixed-integer solution to a system of dual inequalities

$\therefore$ we are looking for finite subeigenvectors of B corresponding to $\lambda = 0$
$\therefore \lambda(B) \leq 0$ is a necessary condition for the solvability of SDI
$\therefore$ the set of all finite solutions to $B \otimes x \leq x$ is

$$V^{**}(B, 0) = \{B^* \otimes z; z \in \mathbb{R}^n\}$$

An application: Bounded mixed-integer solution to a system of dual inequalities

$$(B \otimes x \leq x \text{ and } x \leq u) \iff$$

$$\iff x = B^* \otimes z \leq u, z \in \mathbb{R}^n$$

$$\iff x = B^* \otimes z, \; z \leq (B^*)^{\#} \otimes' u$$

$$\implies x \leq B^* \otimes \left((B^*)^{\#} \otimes' u \right)$$

$\therefore l \leq B^* \otimes \left((B^*)^{\#} \otimes' u \right)$ is necessary and sufficient for the existence of a solution to SDI satisfying $l \leq x \leq u$

An application: Bounded mixed-integer solution to a system of dual inequalities

Algorithm BMISDI

Input: $B \in \mathbb{R}^{n \times n}$, $u, l \in \mathbb{R}^n$ and $J \subseteq N$.

Output: x satisfying BMISDI conditions or an indication that no such vector exists.

$x := u$

$x_j := \lfloor x_j \rfloor$ for $j \in J$

$z := (B^*)^{\#} \otimes' x$, $x := B^* \otimes z$

If $l \nleq x$ then stop (no solution)

If $l \leq x$ and $x_j \in \mathbb{Z}$ for $j \in J$ then stop else go to 2.

BMISDI requires $O(n^3 + n^2 L)$ operations of $+$, max, min, $\leq$ and $\lfloor \cdot \rfloor$, where

$$L = \sum_{j \in J} (u_j - l_j)$$

Finding all eigenvalues: the reduced graph

$A \approx B$ means: A can be obtained from B by a simultaneous permutation of rows and columns

If $A \approx B$ then

$\Lambda(A) = \Lambda(B)$ and

there is a bijection between $V(A, \lambda)$ and $V(B, \lambda)$ for any λ

Frobenius Normal Form (FNF):

$$A \approx \begin{pmatrix} A_{11} & & & & \\ A_{21} & A_{22} & & \varepsilon & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ A_{r1} & A_{r2} & \cdots & \cdots & A_{rr} \end{pmatrix}$$

$A_{11}, ..., A_{rr}$ irreducible

The corresponding partition of N : $N_1, ..., N_r$ ... *classes (of* A)

Reduced digraph Red(A) (partially ordered set)*:*

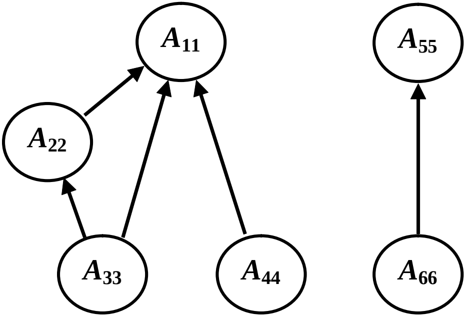
nodes: $1, ..., r$

arcs: $(i, j) : (\exists k \in N_i)(\exists \ell \in N_j) a_{k\ell} > \varepsilon$

$N_i \longrightarrow N_j$ or $i \longrightarrow j$ means: there is a directed path from i to j in Red(A)

Finding all eigenvalues: Reduced digraph

$$\begin{pmatrix} A_{11} & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ * & A_{22} & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ * & * & A_{33} & \varepsilon & \varepsilon & \varepsilon \\ * & \varepsilon & \varepsilon & A_{44} & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & A_{55} & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & * & A_{66} \end{pmatrix} \quad (* \neq \varepsilon)$$



Initial classes: no incoming arcs

Final classes: no outgoing arcs

Finding all eigenvalues: Spectral Theorem

A in an FNF:

$$\begin{pmatrix} A_{11} & & & & \\ A_{21} & A_{22} & & \varepsilon & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ A_{r1} & A_{r2} & \cdots & \cdots & A_{rr} \end{pmatrix}, \; A_{11}, ..., A_{rr} \text{ irreducible}$$

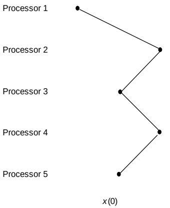
Spectral Theorem (Gaubert, Bapat, 1992):

$$\Lambda(A) = \{\lambda(A_{ii}); \lambda(A_{ii}) \geq \lambda(A_{jj}) \text{ if } j \longrightarrow i\}$$

Corollary: Every matrix has at most n eigenvalues.
i is called *spectral* if $\lambda(A_{ii}) \geq \lambda(A_{jj})$ whenever $j \longrightarrow i$
All real numbers $\lambda \geq \min \Lambda(A)$ are "subeigenvalues", that is
$A \otimes x \leq \lambda \otimes x$ for some $x \neq \varepsilon$.

Part II. Reachability of eigenspaces by matrix orbits

Processors $P_1, ..., P_n$ work interactively and in stages
$x_i(r)$ ... starting time of the r^{th} stage on processor P_i
$(i = 1, \ldots, n; r = 0, 1, ...)$
a_{ij} ... time P_j needs to prepare the component for P_i
$x_i(r+1) = max(x_1(r) + a_{i1}, \ldots, x_n(r) + a_{in})$
$(i = 1, \ldots, n; r = 0, 1, ...)$
$x_i(r+1) = \sum_k^{\oplus} a_{ik} \otimes x_k(r) \quad (i = 1, \ldots, n; r = 0, 1, ...)$
$x(r+1) = A \otimes x(r) \quad (r = 0, 1, \ldots)$
$A : x(0) \rightarrow x(1) \rightarrow x(2) \rightarrow ...$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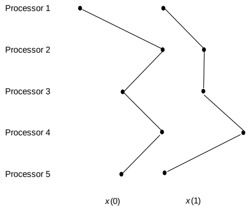
x(0)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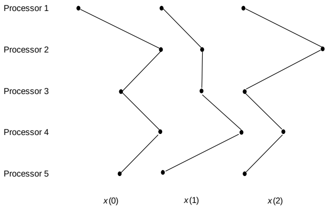
x(0)
x(1)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x(0)
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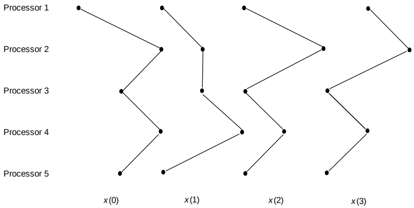
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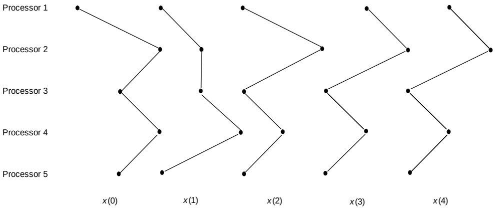
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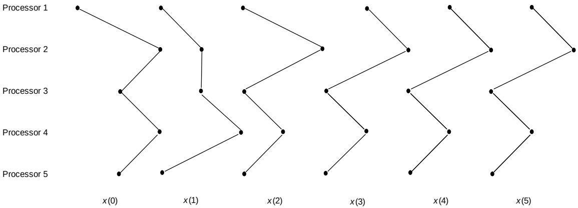
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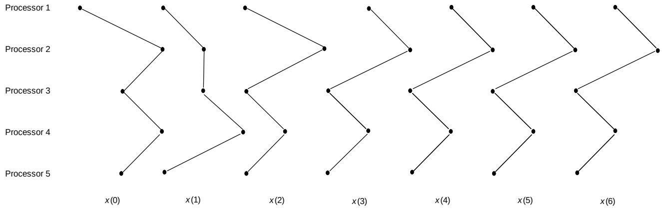
x(5)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x(0)
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x(6)

MPIS: STEADY REGIME

Given $x(0)$, will the MPIS reach a *steady regime* (that is, will it move forward in regular steps)?
Equivalently, is there a λ and an r_0 such that

$$x(r+1) = \lambda \otimes x(r) \quad (r \geq r_0)?$$

$$x(r+1) = A \otimes x(r) \quad (r = 0, 1, \dots)$$

Steady regime is reached if and only if for some λ and r, $x(r)$ is a solution to

$$A \otimes x = \lambda \otimes x$$

Since

$$x(r) = A \otimes x(r-1) = A^2 \otimes x(r-2) = \dots = A^r \otimes x(0),$$

a steady regime is reached if and only if $A^r \otimes x(0)$ "hits" an eigenvector of A for some r.

Reachability

Reachability of an eigenspace: Given $A \in \overline{\mathbb{R}}^{n \times n}$ and an $x \in \overline{\mathbb{R}}^{n}$, $x \neq \varepsilon$, is there a k such that $A^k \otimes x$ is an eigenvector of A?

Various applications - a recent one:
Brackley, Broomhead, Romano, Thiel: Max-Plus Model of Ribosome Dynamics During mRNA Translation, 2011

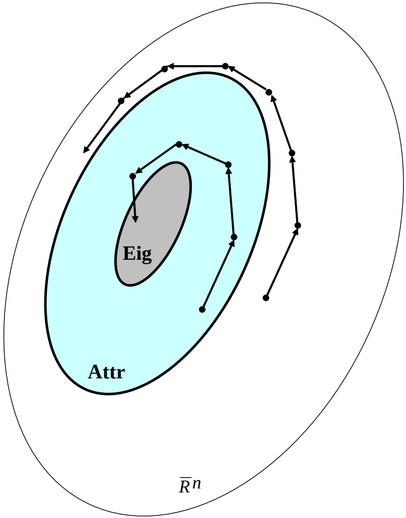
Attraction set

Matrix orbit with starting vector x :
$A \otimes x, A^2 \otimes x, ..., A^k \otimes x, ...$
Attraction set:

$$Attr\,(A) = \left\{x; (\exists k)\, A^k \otimes x \in V(A)\right\}$$

$$V(A) \subseteq attr\,(A)$$

Reachability



Cyclicity of a matrix

Cyclicity (index of imprimitivity) of a strongly connected digraph $=$ g.c.d. of the lengths of its cycles

Cyclicity of a digraph $=$ l.c.m. of cyclicities of its SCC

Let $A \in \overline{\mathbb{R}}^{n \times n}$

C_A ... *critical digraph* of A

Cyclicity of a matrix A : $\sigma(A) =$ cyclicity of C_A

A is *primitive* if $\sigma(A) = 1$

Cyclicity Theorem (Cohen et al 1985)

For every irreducible matrix $A \in \overline{\mathbb{R}}^{n \times n}$ *the cyclicity of* A *is the period of* A, *that is, the smallest natural number* p *for which there is an integer* $T(A)$ *such that*

$$A^{r+p} = (\lambda(A))^p \otimes A^r$$

for every $r \geq T(A)$

$T(A)$ ... *transient* of A

Cyclicity Theorem

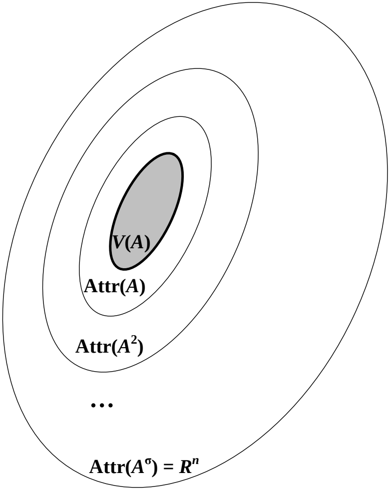
For A irreducible:

$$A^{r+\sigma} = (\lambda(A))^{\sigma} \otimes A^{r}, r \geq T(A)$$

For any A (irreducible or not): $\lambda\left(A^{k}\right) = (\lambda(A))^{k}$ for every integer $k \geq 0$
$\therefore$ For A irreducible:

$$A^{\sigma} \otimes (A^{r} \otimes x) = \lambda(A^{\sigma}) \otimes (A^{r} \otimes x), \quad r \geq T(A)$$

Corollary
If A is irreducible then $(\forall x \neq \varepsilon)\, A^{r} \otimes x \in V(A^{s})$ for some r and $s \leq \sigma(A)$
Reachability asks about $s = 1$



V(A)
Attr(A)
Attr(A²)
...
Attr(A^σ) = R^n

Reachability for irreducible matrices

Theorem (Nachtigall, 1997): Let $A \in \overline{\mathbb{R}}^{n \times n}$ be irreducible. Critical rows and columns of A^r are periodic for $r \geq n^2$, that is for all $(i, j) \in (N_c(A) \times N) \cup (N \times N_c(A))$ we have:

$$a_{ij}^{(r+\sigma)} = (\lambda(A))^{\sigma} \otimes a_{ij}^{(r)}$$

Theorem (Sergeev, 2009): Let $A \in \overline{\mathbb{R}}^{n \times n}$ be irreducible and definite. Then for every $r \geq T(A)$ and $k = 1, \ldots, n$ coefficients $\alpha_i \in \overline{\mathbb{R}}$ $(i \in N_c(A))$ such that

$$A_{k\cdot}^{r} = \sum\nolimits_{i \in N_c(A)}^{\oplus} \alpha_i \otimes A_{i\cdot}^{r}$$

can be found in $O(n^3)$ time.

Reachability for irreducible matrices

Corollary: If A is irreducible and definite then A^r for any $r \geq T(A)$ can be found in $O\left(n^3 \log n\right)$ time (but not r)

$\therefore$ Reachability of $V(A^s)$ for any s for A irreducible and definite, and any x can be checked in $O\left(n^3 \log n\right)$ time

Reachability for reducible matrices

REACHABILITY: *Given* $A \in \overline{\mathbb{R}}^{n\times n}$, *its eigenvalue* λ *and* $x \in \overline{\mathbb{R}}^{n}$, *is there a* k *such that* $z = A^k \otimes x$ *is an eigenvector with eigenvalue* λ*? That is* $A \otimes z = \lambda \otimes z$.

Now A reducible - in a Frobenius Normal Form:

$$\begin{pmatrix} A_{11} & & & & \\ A_{21} & A_{22} & & \varepsilon & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ A_{r1} & A_{r2} & \cdots & \cdots & A_{rr} \end{pmatrix}, A_{11}, ..., A_{rr} \text{ irreducible}$$

$R = \{1, ..., r\}$

A_{ii} irreducible may be 1×1 matrix (ε) ... "trivial" - **exclude at first**

Reachability for reducible matrices

For any $x \in \overline{\mathbb{R}}^n$ denote

$$\begin{aligned} J(x) &= \{j \in R; x[N_j] \neq \varepsilon\}, \\ C(x) &= \{i \in R; (\exists j \in J(x))\, N_i \longrightarrow N_j\}. \end{aligned}$$

$N_i, i \in J(x)$ is *final* in $C(x)$ if $N_i \longrightarrow N_j$ is not true for any $j \in J(x)$, $j \neq i$.
If $y = A \otimes x$ then

$$y[N_i] = \sum\nolimits_{j \in R}^{\oplus} A_{ij} \otimes x[N_j] \text{ for every } i \in R.$$

Suppose N_i is final in $C(x)$ then

$$y[N_i] = A_{ii} \otimes x[N_i]$$

Reachability for reducible matrices

If N_i is final in $C(x)$ then

$$y[N_i] = A_{ii} \otimes x[N_i]$$

[If $B \neq \varepsilon$ is irreducible and $v \neq \varepsilon$ then $B \otimes v \neq \varepsilon$]
$\therefore y[N_i] \neq \varepsilon$ (since $A_{ii} \neq \varepsilon$)
Proposition: Final classes in $C(x)$ and $C(y)$ coincide
Corollary: If $A \otimes z = \lambda \otimes z$ and $z = A^k \otimes x$ for some k then the final classes in $C(x)$ and $C(z)$ coincide and

$$z[N_i] = A_{ii}^k \otimes x[N_i]$$

for any final class N_i in $C(x)$

Reachability for reducible matrices

$A \otimes z = \lambda \otimes z$ blockwise:

$$\sum\nolimits_{j\in R}^{\oplus} A_{ij} \otimes z[N_j] = \lambda \otimes z[N_i] \text{ for every } i \in R$$

If N_i is final in $C(z)$:

$$A_{ii} \otimes z[N_i] = \lambda \otimes z[N_i]$$

$\therefore$ If N_i final in $C(x)$ then $x[N_i] \in attr(A_{ii})$
This is can be checked in $O(n^3 \log n)$ time
We may assume that a periodic regime for all final classes has been reached

Reachability for reducible matrices

If $A \otimes z = \lambda \otimes z$, and $z[N_i] \neq \varepsilon$ then
$z[N_i]$ is finite,
$\lambda(A_{ii}) \leq \lambda$ and
$N_i \longrightarrow N_j$, where $\lambda(A_{jj}) = \lambda$
in particular, $\lambda(A_{ii}) = \lambda$ if N_i is final in $C(z)$
We have already seen that $x[N_i] \neq \varepsilon \Longrightarrow z[N_i] \neq \varepsilon$ if $z = A^k \otimes x$
so a necessary reachability condition is:

$$x[N_i] \neq \varepsilon \Longrightarrow \lambda(A_{ii}) \leq \lambda$$

Reachability for reducible matrices

From now on assume that a periodic regime for all final classes has been reached $(z = x)$

We may also assume that $\lambda = 0$

$\therefore$ for any final class $x[N_i]$:

$$A_{ii} \otimes x[N_i] = x[N_i]$$

Reachability for reducible matrices

Checking the non-final classes - explanation for $r = 2$:

$$A = \begin{pmatrix} A_{11} & \varepsilon \\ A_{21} & A_{22} \end{pmatrix}, x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$A_{11} \otimes x_1 = x_1$
Without loss of generality: $A_{21} \neq \varepsilon$ and $x_1 \neq \varepsilon$
$\therefore \lambda \left(A_{22} \right) \leq 0$
Denote $x^0 = x$ and

$$x^k = A^k \otimes x^0 = \begin{pmatrix} x_1^k \\ x_2^k \end{pmatrix}$$

Since $x_1^k = x_1^0$ for every k we only need to check whether x_2^k is stationary ($\otimes$ omitted):

$$x_2^k = A_{21} x_1^0 \oplus A_{22} x_2^{k-1} = \ldots$$

Since $x_1^k = x_1^0$ for every k we only need to check whether x_2^k is stationary:

$$\begin{aligned} x_2^k &= A_{21}x_1^0 \oplus A_{22}x_2^{k-1} \\ &= A_{21}x_1^0 \oplus A_{22}\left(A_{21}x_1^0 \oplus A_{22}x_2^{k-2}\right) \\ &\ldots \\ &= \left(I \oplus \ldots \oplus A_{22}^{k-1}\right)A_{21}x_1^0 \oplus A_{22}^k x_2^0 \\ &= A_{22}^* A_{21}x_1^0 \oplus A_{22}^k x_2^0 \end{aligned}$$

$v = A_{22}^* A_{21} x_1^0$ ... constant *finite* vector for $k \geq n-1$
If $\lambda\left(A_{22}\right) < 0$ then $A_{22}^k x_2^0 \longrightarrow -\infty$ and so $x_2^k = A_{22}^* A_{21} x_1^0$ for k large
If $\lambda\left(A_{22}\right) = 0$:
A_{22} is irreducible and definite
periodic regime of A_{22} can be found in polynomial time

Reachability for reducible matrices

For $k \geq T(A_{11})$ (unknown):

$$A^k \otimes x = \begin{pmatrix} A_{11}^k & \varepsilon \\ A_{22}^* A_{21} & A_{22}^k \end{pmatrix} \otimes x$$

$A_{11}^{k+1} \otimes x_1 = A_{11}^k \otimes x_1$

A_{11}^k can be found in polynomial time (for any $k \geq T(A_{11})$)
$A_{22}^{k+\sigma} \otimes x_2 = A_{22}^k \otimes x_2$ where $\sigma = \sigma(A_{22})$, $k \geq T(A_{22})$
A_{22}^k can be found in polynomial time (for any $k \geq T(A_{22})$)
$\therefore A^{k+s} \otimes x = A^k \otimes x$ for some $s \leq \sigma$ and
$k \geq \max(T(A_{11}), T(A_{22}), n)$
A^k can be found in polynomial time (for any k large)

Reachability for reducible matrices

Lemma: *If there exist natural numbers s and T such that*

$A^{t+s} \otimes x = A^t \otimes x$ *for every* $t \geq T$ *then*

$$A^{r+1} \otimes x = A^r \otimes x$$

for a natural number $r \geq T$ *if and only if*

$$A^{k+1} \otimes x = A^k \otimes x$$

for every natural number $k \geq T$.
Checking reachability thus reduces to checking

$$A^{k+1} \otimes x = A^k \otimes x$$

for $k \geq \max\left(T(A_{11}), T(A_{22}), n\right)$.

The role of trivial blocks

$$\begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon \end{pmatrix} : \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} \varepsilon \\ 0 \\ 0 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} \varepsilon \\ \varepsilon \\ 0 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} \varepsilon \\ \varepsilon \\ \varepsilon \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} \varepsilon \\ \varepsilon \\ \varepsilon \\ \varepsilon \end{pmatrix} \longrightarrow \dots$$

$$\begin{pmatrix} 0 & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon \end{pmatrix} : \begin{pmatrix} 0 \\ \varepsilon \\ \varepsilon \\ \varepsilon \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 0 \\ \varepsilon \\ \varepsilon \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ \varepsilon \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \longrightarrow \dots$$

Reachability for reducible matrices

Proposition: Let $z^{(k)} = A^k \otimes x, k = 1, 2, \dots$. If N_i is trivial (that is $A_{ii} = (\varepsilon)$) then
either $z^{(k)}[N_i] = \varepsilon$ for all $k \geq 2n$
or $z^{(k)}[N_i] \neq \varepsilon$ for all $k \geq 2n$
Proposition: For every $k \geq 2n$ every i the class $z^{(k+1)}[N_i]$ is final if and only if $z^{(k)}[N_i]$ is final.
Corollary: For solving REACHABILITY it is sufficient to first move $x \longrightarrow A^{2n} \otimes x$

Strongly and weakly stable matrices

$V(A) \subseteq attr(A) \subseteq \overline{\mathbb{R}}^n - \{\varepsilon\}$
Two extremes:
$attr(A) = \overline{\mathbb{R}}^n - \{\varepsilon\}$ ... *A strongly stable* (robust)
$attr(A) = V(A)$ ... *A weakly stable*

Strong stability (robustness)

If A is irreducible and primitive then by the Cyclicity Theorem:
$A^{k+1} = \lambda(A) \otimes A^k$ for k large
$A^{k+1} \otimes x = \lambda(A) \otimes A^k \otimes x$ for k large and any $x \in \overline{\mathbb{R}}^n$
A irreducible: A is robust $\Longleftrightarrow$ A is primitive
Robustness criterion for reducible matrices (PB & S.Gaubert & RACG 2009):
A with FNF classes $N_1, ...N_r$ and no ε column is robust if and only if
All nontrivial classes are primitive and spectral
$(\forall i, j)$ If N_i, N_j are non-trivial, $N_i \nrightarrow N_j$ and $N_j \nrightarrow N_i$ then

$$\lambda(A_{ii}) = \lambda(A_{jj})$$

Strong stability (robustness)

Reduced digraph of a robust matrix with $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$:

Weakly stable matrices

A weakly stable: $attr(A) = V(A)$

Let A be irreducible

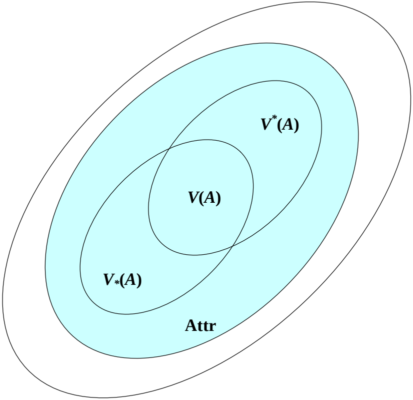
$V(A) = \{x \in \overline{\mathbb{R}}^n; A \otimes x = \lambda(A) \otimes x, x \neq \varepsilon\}$ ... *eigenvectors*

$V_*(A) = \{x \in \overline{\mathbb{R}}^n; A \otimes x \leq \lambda(A) \otimes x, x \neq \varepsilon\}$ ... *subeigenvectors*

$V^*(A) = \{x \in \overline{\mathbb{R}}^n; A \otimes x \geq \lambda(A) \otimes x, x \neq \varepsilon\}$ ... *supereigenvectors*

Weakly stable matrices

$V(A) \subseteq V_*(A) \subseteq Attr(A)$
$V(A) \subseteq V^*(A) \subseteq Attr(A)$



A weakly stable $\Longrightarrow V(A) = V^*(A) = V_*(A) = Attr(A)$

Weakly stable matrices

Let A be irreducible.
A is weakly stable $\Longleftrightarrow$ C_A is a Hamilton cycle in D_A.

$$\begin{pmatrix} & * & & & \\ & & * & & \\ & & & * & \\ & & & & * \\ * & & & & \end{pmatrix}$$

Weakly stable matrices

A is weakly stable if and only if every spectral class of A is initial and weakly stable

Visualisation

$A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$ is called *visualised* if

$a_{ij} \leq 0$ for all i, j

$a_{ij} = 0$ if $(i, j) \in E_c$

If $\lambda(A) > \varepsilon$ and x is a finite eigenvector then $B = X^{-1} \otimes A_\lambda \otimes X$ is visualised, where $X = diag(x_1, .., x_n)$

There is a bijection between $V(A)$ and $V(B)$ and $\lambda(B) = \lambda(A_\lambda) = 0$

$\therefore$ There is no loss of generality to assume that A is visualised and definite.

P. Butkovic: Max-linear Systems: Theory and Algorithms (Springer Monographs in Mathematics, Springer-Verlag 2010)

For A irreducible and definite:

$$A^{r+\sigma} = A^r \text{ for all } r \geq T(A)$$

Corollary 1: Let A be irreducible and definite and $r \geq T(A)$. Then $A^r \otimes x = A^{r+p} \otimes x$ is equivalent to its critical subsystem for $r \geq n^2$.

Example: If

$$A = \begin{pmatrix} -2 & \boxed{1} & -3 \\ 3 & 0 & \boxed{3} \\ \boxed{5} & 2 & 1 \end{pmatrix}$$

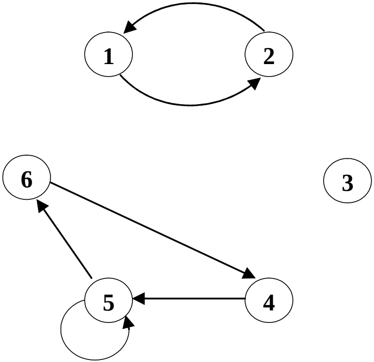
then

$$\begin{aligned} \lambda(A) &= \max\{-2, 0, 1, 2, 1, 5/2, 3, 2/3\} = 3 \\ \sigma &= (1, 2, 3) \text{ is critical} \end{aligned}$$

Eigenproblem: The principal eigenvalue and eigenvectors

$$A = \begin{pmatrix} 7 & \boxed{9} & 5 & 5 & 3 & 7 \\ \boxed{7} & 5 & 2 & 7 & 0 & 4 \\ 8 & 0 & 3 & 3 & 8 & 0 \\ 7 & 2 & 5 & 7 & \boxed{9} & 5 \\ 4 & 2 & 6 & 6 & 8 & \boxed{8} \\ 3 & 0 & 5 & \boxed{7} & 1 & 2 \end{pmatrix}, \quad \lambda(A) = 8$$

Eigenproblem: The principal eigenvalue and eigenvectors



Critical cycles: $(1, 2, 1)$, $(5, 5)$, $(4, 5, 6, 4)$

Node sets of all strongly connected components:

$\{1, 2\}$, $\{3\}$, $\{4, 5, 6\}$

Three strongly connected components, one of them trivial

$N_c = \{1, 2, 4, 5, 6\}$

$$\begin{pmatrix} -2 & 2 & 2 \\ -5 & -3 & -2 \\ \varepsilon & \varepsilon & 3 \\ -3 & -3 & 2 \\ 1 & 4 & \varepsilon \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 1 \\ 0 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} -5 & \boxed{-1} & -1 \\ \boxed{-3} & \boxed{-1} & 0 \\ \varepsilon & \varepsilon & \boxed{2} \\ \boxed{-3} & -3 & \boxed{2} \\ -4 & \boxed{-1} & \varepsilon \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$M_1 = \{2, 4\}$, $M_2 = \{1, 2, 5\}$, $M_3 = \{3, 4\}$

$\overline{x} = (3, 1, -2)^T$ is a solution since $\bigcup_{j=1,2,3} M_j = M$

$M_2 \cup M_3 = M$ hence the solution set is

$$\left\{ (x_1, x_2, x_3)^T \in \overline{\mathbb{R}}^3; x_1 \leq 3, x_2 = 1, x_3 = -2 \right\}$$

$$\underbrace{\begin{pmatrix} 7 & \boxed{9} & 5 & 5 & 3 & 7 \\ \boxed{7} & 5 & 2 & 7 & 0 & 4 \\ 8 & 0 & 3 & 3 & 8 & 0 \\ 7 & 2 & 5 & 7 & \boxed{9} & 5 \\ 4 & 2 & 6 & 6 & 8 & \boxed{8} \\ 3 & 0 & 5 & \boxed{7} & 1 & 2 \end{pmatrix}}_{A}, \quad \lambda(A) = 8$$

$$\underbrace{\begin{pmatrix} 7 & \boxed{9} & 5 & 5 & 3 & 7 \\ \boxed{7} & 5 & 2 & 7 & 0 & 4 \\ 8 & 0 & 3 & 3 & 8 & 0 \\ 7 & 2 & 5 & 7 & \boxed{9} & 5 \\ 4 & 2 & 6 & 6 & 8 & \boxed{8} \\ 3 & 0 & 5 & \boxed{7} & 1 & 2 \end{pmatrix}}_{A} \xrightarrow{-8} \underbrace{\begin{pmatrix} -1 & \boxed{1} & -3 & -3 & -5 & -1 \\ \boxed{-1} & -3 & -6 & -1 & -8 & -4 \\ 0 & -8 & -5 & -5 & 0 & -8 \\ -1 & -6 & -3 & -1 & \boxed{1} & -3 \\ -4 & -6 & -2 & -2 & 0 & \boxed{0} \\ -5 & -8 & -5 & \boxed{-1} & -7 & -6 \end{pmatrix}}_{A_\lambda}$$

$$\underbrace{\begin{pmatrix} 7 & \boxed{9} & 5 & 5 & 3 & 7 \\ \boxed{7} & 5 & 2 & 7 & 0 & 4 \\ 8 & 0 & 3 & 3 & 8 & 0 \\ 7 & 2 & 5 & 7 & \boxed{9} & 5 \\ 4 & 2 & 6 & 6 & 8 & \boxed{8} \\ 3 & 0 & 5 & \boxed{7} & 1 & 2 \end{pmatrix}}_{A} \xrightarrow{-8} \underbrace{\begin{pmatrix} -1 & \boxed{1} & -3 & -3 & -5 & -1 \\ \boxed{-1} & -3 & -6 & -1 & -8 & -4 \\ 0 & -8 & -5 & -5 & 0 & -8 \\ -1 & -6 & -3 & -1 & \boxed{1} & -3 \\ -4 & -6 & -2 & -2 & 0 & \boxed{0} \\ -5 & -8 & -5 & \boxed{-1} & -7 & -6 \end{pmatrix}}_{A_\lambda}$$

$$\underbrace{\begin{pmatrix} \boxed{0} & 1 & -1 & 0 & 1 & 1 \\ -1 & \boxed{0} & 2 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ -1 & 0 & -1 & \boxed{0} & 1 & 1 \\ -2 & -1 & -2 & -1 & \boxed{0} & 0 \\ -2 & -1 & -2 & -1 & 0 & \boxed{0} \end{pmatrix}}_{A_\lambda^+}$$

$$\underbrace{\begin{pmatrix} 7 & \boxed{9} & 5 & 5 & 3 & 7 \\ \boxed{7} & 5 & 2 & 7 & 0 & 4 \\ 8 & 0 & 3 & 3 & 8 & 0 \\ 7 & 2 & 5 & 7 & \boxed{9} & 5 \\ 4 & 2 & 6 & 6 & 8 & \boxed{8} \\ 3 & 0 & 5 & \boxed{7} & 1 & 2 \end{pmatrix}}_{A} \xrightarrow{-8} \underbrace{\begin{pmatrix} -1 & \boxed{1} & -3 & -3 & -5 & -1 \\ \boxed{-1} & -3 & -6 & -1 & -8 & -4 \\ 0 & -8 & -5 & -5 & 0 & -8 \\ -1 & -6 & -3 & -1 & \boxed{1} & -3 \\ -4 & -6 & -2 & -2 & 0 & \boxed{0} \\ -5 & -8 & -5 & \boxed{-1} & -7 & -6 \end{pmatrix}}_{A_\lambda}$$

$$\underbrace{\begin{pmatrix} \boxed{0} & 1 & -1 & 0 & 1 & 1 \\ -1 & \boxed{0} & 2 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ -1 & 0 & -1 & \boxed{0} & 1 & 1 \\ -2 & -1 & -2 & -1 & \boxed{0} & 0 \\ -2 & -1 & -2 & -1 & 0 & \boxed{0} \end{pmatrix}}_{A_\lambda^+} \longrightarrow \begin{pmatrix} 0 & . & . & 0 & . & . \\ -1 & . & . & -1 & . & . \\ 0 & . & . & 0 & . & . \\ -1 & . & . & 0 & . & . \\ -2 & . & . & -1 & . & . \\ -2 & . & . & -1 & . & . \end{pmatrix}$$

Eigenproblem: The principal eigenvalue and eigenvectors

$$A = \begin{pmatrix} 0 & 3 & & \\ 1 & -1 & & \\ & & 2 & \\ & & & 1 \end{pmatrix}, \text{ blank} = \varepsilon$$

$\lambda(A) = 2$

$N_c = \{1, 2, 3\}$

$1 \sim 2$

$\dim(A) = 2$

$$A_\lambda^+ = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & \\ & & & -1 \end{pmatrix}$$

A basis of the principal eigenspace is e.g.

$$\left\{ g_2 = (1, 0, \varepsilon, \varepsilon)^T, g_3 = (\varepsilon, \varepsilon, 0, \varepsilon)^T \right\}$$

Finding all eigenvalues

$$A = \begin{pmatrix} \boxed{\begin{matrix} 0 & 3 \\ 5 & 1 \end{matrix}} & & & & & \\ & & \boxed{4} & & & \\ & & 0 & \boxed{\begin{matrix} 3 & 1 \\ -1 & 2 \end{matrix}} & & \\ & & & 1 & & \boxed{5} \end{pmatrix} \quad (\text{blank} = \varepsilon)$$

$\lambda(A_{11}) = 4, \lambda(A_{22}) = 4, \lambda(A_{33}) = 3, \lambda(A_{44}) = 5, r = 4$
$\lambda(A) = 5$
$\Lambda(A) = \{4, 5\}$
N_1, N_4 are spectral (N_2 is not)

Principal eigenspace

$$\lambda(A) > \varepsilon$$
$$A_\lambda = (\lambda(A))^{-1} \otimes A$$
$$A^+ = \quad A \oplus A^2 \oplus \ldots \oplus A^{n-1} \oplus A^n$$

$$A \longrightarrow A_\lambda$$

Principal eigenspace

$\lambda(A) > \varepsilon$
$A_\lambda = (\lambda(A))^{-1} \otimes A$
$A^+ = \quad A \oplus A^2 \oplus \ldots \oplus A^{n-1} \oplus A^n$

$A \longrightarrow A_\lambda \longrightarrow (A_\lambda)^+$

Principal eigenspace

$$\lambda(A) > \varepsilon$$
$$A_\lambda = (\lambda(A))^{-1} \otimes A$$
$$A^+ = \quad A \oplus A^2 \oplus \ldots \oplus A^{n-1} \oplus A^n$$

$$A \longrightarrow A_\lambda \longrightarrow (A_\lambda)^+ \quad \text{(briefly } A_\lambda^+\text{)}$$

Principal eigenspace

If $\lambda(A) > \varepsilon$ then every column of A_λ^+ with zero diagonal entry is an eigenvector of A with corresponding eigenvalue $\lambda(A)$ (*principal eigenvector*)

An essentially unique basis of $V(A, \lambda(A))$ (the *principal eigenspace*) can be obtained by taking exactly one principal eigenvector of A for each equivalence class in $(N_c, \sim)$

If $A_\lambda^+ = (g_1, \ldots, g_n)$ then $i \sim j$ if and only if $g_i = \alpha \otimes g_j, \alpha \in \mathbb{R}$

If A is irreducible then $V(A) = V(A, \lambda(A))$ and $V(A) \subseteq \mathbb{R}^n$