Green’s $J$-order and the rank of tropical matrices

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Let $T = \mathbb{R} \cup \{-\infty\}$ and define two binary operations on $T$ by

$$a \oplus b := \max(a, b), \quad \text{and} \quad a \otimes b := a + b,$$

for all $a, b \in T$ (where $a \oplus -\infty = -\infty \oplus a = a$ and $a \otimes -\infty = -\infty \otimes a = -\infty$).
The tropical semiring

Let \( T = \mathbb{R} \cup \{-\infty\} \) and define two binary operations on \( T \) by
\[
a \oplus b := \max(a, b), \quad \text{and} \quad a \otimes b := a + b,
\]
for all \( a, b \in T \) (where \( a \oplus -\infty = -\infty \oplus a = a \) and \( a \otimes -\infty = -\infty \otimes a = -\infty \)).

- \((T, \oplus)\) is a commutative monoid with identity element \(-\infty\);
- \((T, \otimes)\) is a (commutative) monoid with identity element 0;
- \( \otimes \) distributes over \( \oplus \);
- \(-\infty\) is an absorbing element with respect to \( \otimes \);
- For all \( a \in T \) we have \( a \oplus a = a \).

We say that \( T \) is a (commutative) idempotent semiring. It is often referred to as the max-plus or tropical semiring.
Motivation

The tropical semiring has applications in diverse areas such as...

- analysis of discrete event systems
- combinatorial optimisation and scheduling problems
- formal languages and automata
- statistical inference
- algebraic geometry...

Typically problems in application areas involve finding solutions to a system of linear equations over the tropical semiring.

Thus it is natural to consider matrices with entries in the tropical semiring...
Consider the set $M_n(\mathbb{T})$ of all $n \times n$ matrices with entries in $\mathbb{T}$. The operations $\oplus$ and $\otimes$ can be extended to such matrices in the usual way:

\[
(A \oplus B)_{i,j} = A_{i,j} \oplus B_{i,j}, \text{ for all } A, B \in M_n(\mathbb{T})
\]

\[
(A \otimes B)_{i,j} = \bigoplus_{k=1}^{l} A_{i,k} \otimes B_{k,j}, \text{ for all } A, B \in M_n(\mathbb{T}).
\]

We study the multiplicative semigroup $(M_n(\mathbb{T}), \otimes)$. 
Tropical convex sets

We write $\mathbb{T}^n$ to denote the set of all $n$-tuples $x = (x_1, \ldots, x_n)$ with $x_i \in \mathbb{T}$ and extend $\oplus$ to $\mathbb{T}^n$ componentwise:

$$(x \oplus y)_i = x_i \oplus y_i.$$ 

We also define a scaling action of $\mathbb{T}$ on $\mathbb{T}^n$:

$$(\lambda \otimes x)_i, \ldots, x_n) = \lambda \otimes x_i, \text{ for all } \lambda \in \mathbb{T}.$$ 

A tropical convex set $X$ in $\mathbb{T}^n$ is a subset that is closed under $\oplus$ and scaling. We say that a subset $V \subseteq X$ is a generating set for $X$ if every element of $X$ can be written as a tropical linear combination of finitely many elements of $V$. 
Let $A, B \in M_n(\mathbb{T})$.

(1) $A \mathcal{L} B \iff \text{row space of } A = \text{row space of } B$.

(2) $A \mathcal{R} B \iff \text{col. space of } A = \text{col. space of } B$.

(3) $A \mathcal{H} B \iff \text{row space of } A = \text{row space of } B$ and 
    \text{col. space of } A = \text{col. space of } B.

(4) $A \mathcal{D} B \iff \text{row space of } A \cong \text{row space of } B$
    \iff \text{col. space of } A \cong \text{col. space of } B
    \iff \text{Recent result of Hollings and Kambites.}$

(Note: The row space need not be linearly isomorphic to the column space.)

We describe Green’s $\mathcal{J}$-order (and hence the corresponding $\mathcal{J}$-relation).
Is $\mathcal{D} = \mathcal{J}$?

**Example** $AJB$ but $A\emptyset B$.

$$A = \begin{pmatrix} -\infty & 0 & 1 & -\infty \\ -\infty & -\infty & 1 & -\infty \\ 0 & 0 & 0 & -\infty \\ -\infty & -\infty & -\infty & -\infty \end{pmatrix}, \quad B = \begin{pmatrix} -\infty & 0 & 1 & 1 \\ -\infty & -\infty & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -\infty & -\infty & -\infty & -\infty \end{pmatrix}$$

- Claim there exist matrices $P, Q, R, S \in M_4(\mathbb{T})$ such that $A = PBQ$ and $B = RAS$.
- It is easy to check that $C(A)$ can be generated by three elements, whilst $C(B)$ cannot be generated by fewer than four elements.
- Thus the column spaces $C(A)$ and $C(B)$ are not linearly isomorphic and hence $A\emptyset B$. 

Theorem. For the subsemigroup of matrices without $-\infty$ entries we have that $\mathcal{D} = \mathcal{J}$. 
Is \( D = J \)?

**Example** \( AJB \) but \( A \not\sim B \).

\[
A = \begin{pmatrix}
-\infty & 0 & 1 & -\infty \\
-\infty & -\infty & 1 & -\infty \\
0 & 0 & 0 & -\infty \\
-\infty & -\infty & -\infty & -\infty
\end{pmatrix},
B = \begin{pmatrix}
-\infty & 0 & 1 & 1 \\
-\infty & -\infty & 1 & 0 \\
0 & 0 & 0 & 0 \\
-\infty & -\infty & -\infty & -\infty
\end{pmatrix}
\]

- Claim there exist matrices \( P, Q, R, S \in \text{M}_4(\mathbb{T}) \) such that \( A = PBQ \) and \( B = RAS \).
- It is easy to check that \( C(A) \) can be generated by three elements, whilst \( C(B) \) cannot be generated by fewer than four elements.
- Thus the column spaces \( C(A) \) and \( C(B) \) are not linearly isomorphic and hence \( A \not\sim B \).

**Theorem.** For the subsemigroup of matrices without \( -\infty \) entries we have that \( D = J \).
Green’s $\mathcal{J}$-order on $M_n(\mathbb{T})$

**Theorem.** Let $A, B \in M_n(\mathbb{T})$. Then the following are equivalent.

(i) $A \leq_{\mathcal{J}} B$;

(ii) There is a $\mathbb{T}$-linear convex set $X$ such that the row space of $A$ embeds linearly into $X$ and the row space of $B$ surjects linearly onto $X$;

(iii) There is a $\mathbb{T}$-linear convex set $Y$ such that the col. space of $A$ embeds linearly into $Y$ and the col. space of $B$ surjects linearly onto $Y$. 
Lemma. Let $A, B \in M_n(\mathbb{T})$. Then the following are equivalent.

(i) $R(A)$ embeds linearly into $R(B)$;
(ii) $C(B)$ surjects linearly onto $C(A)$;
(iii) There exists $C \in M_n(\mathbb{T})$ with $ARC \leq L B$. 
Lemma. Let $A, B \in M_n(\mathbb{T})$. Then the following are equivalent.

(i) $R(A)$ embeds linearly into $R(B)$;
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Lemma. Let $A, B \in M_n(\mathbb{T})$. Then the following are equivalent.

(i) $C(A)$ embeds linearly into $C(B)$;
(ii) $R(B)$ surjects linearly onto $R(A)$;
(iii) There exists $C \in M_n(\mathbb{T})$ with $ALC \leq_R B$. 
The rank of a tropical matrix

There are several (non-equivalent) notions of the rank of a tropical matrix. We define three such here...

\[
\text{factor rank}(A) = \text{the minimum } k \text{ such that } A \text{ can be factored as } A = CR \text{ where } C \text{ is } n \times k \text{ and } R \text{ is } k \times n
\]

\[
\text{det rank}(A) = \text{the maximum } k \text{ such that } A \text{ has a } k \times k \text{ minor } M \text{ with } |M|^+ \neq |M|^-
\]

\[
\text{tropical rank}(A) = \text{the maximum } k \text{ such that } A \text{ has a } k \times k \text{ minor } M \text{ where the max. is achieved twice in the permanent of } M.
\]
Let $A, B \in M_n(\mathbb{T})$. Then it is known that

\[
\begin{align*}
\text{factor rank}(AB) & \leq \min(\text{factor rank}(A), \text{factor rank}(B)) \\
\text{det rank}(AB) & \leq \min(\text{det rank}(A), \text{det rank}(B)) \\
\text{tropical rank}(AB) & \leq \min(\text{tropical rank}(A), \text{tropical rank}(B))
\end{align*}
\]

from which it follows easily that...
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\text{tropical rank}(AB) & \leq \min(\text{tropical rank}(A), \text{tropical rank}(B))
\end{align*}
\]

from which it follows easily that...

**Theorem.** The factor rank, det rank and tropical rank are all $\mathcal{J}$-class invariants in $M_n(\mathbb{T})$. 