Idempotent tropical matrices

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Recall the tropical semifield, $\mathbb{FT} = (\mathbb{R}, \oplus, \otimes)$, where

$$a \oplus b := \max(a, b), \quad a \otimes b := a + b.$$ 

Let $M_n(\mathbb{FT})$ denote the set of all $n \times n$ matrices over $\mathbb{FT}$, with multiplication $\otimes$ defined as you would expect:

$$(A \otimes B)_{i,j} = \bigoplus_{k=1}^{n} A_{i,k} \otimes B_{k,j}, \text{ for all } A, B \in M_n(\mathbb{FT}).$$

It is easy to see that $(M_n(\mathbb{FT}), \otimes)$ is a semigroup.
Tropical polytopes

We write $\mathbb{FT}^n$ to denote the set of all $n$-tuples $x = (x_1, \ldots, x_n)$ with $x_i \in \mathbb{FT}$. Then $\mathbb{FT}^n$ has the structure of an $\mathbb{FT}$-module:

$$(x \oplus y)_i = x_i \oplus y_i, \quad (\lambda \otimes x)_i = \lambda \otimes x_i,$$

for all $x, y \in \mathbb{FT}^n$ and all $\lambda \in \mathbb{FT}$.

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A **tropical polytope** is a finitely generated submodule of $\mathbb{FT}^n$.

Let $A \in M_n(\mathbb{FT})$. We define the **row space** $R(A) \subseteq \mathbb{FT}^n$ to be the tropical polytope generated by the rows of $A$.

Similarly, we define the **column space** $C(A) \subseteq \mathbb{FT}^n$ to be the tropical polytope generated by the columns of $A$.
Dimensions of tropical polytopes

Let $X \subseteq \mathbb{FT}^n$ be a tropical polytope.

- The **tropical dimension** of $X$ is the maximum topological dimension of $X$ regarded as a subset of $\mathbb{R}^n$.
- We say that $X$ has **pure tropical dimension** $k$ if every open subset of $X$ has topological dimension $k$. 
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- The **generator dimension** of $X$ is the minimum cardinality of a generating set for $X$.
- The **dual dimension** of $X$ is the minimum $k$ such that $X$ embeds linearly into $FT^k$. 

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Theorem [IJK] Let $X \subseteq \mathbb{FT}^n$ be a tropical polytope.

There is a positive integer $k$ such that $X$ has pure tropical dimension $k$, generator dimension $k$ and dual dimension $k$ if and only if $X$ is the column space of an idempotent if and only if $X$ is projective as an $\mathbb{FT}$-module.
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- If this common dimension is $k$ we say that $X$ is a **projective $k$-polytope**.

- Moreover, if $E$ is an idempotent with $X = C(E)$, we say that $E$ has **rank** $k$. (Note: $1 \leq \text{rank}(E) \leq n$.)
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Maximal subgroups

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**Theorem [IJK]**

Let $E$ be an idempotent in $M_n(\mathbb{F}_T)$ and let $H_E$ denote the maximal subgroup containing $E$. Then

- $H_E$ is isomorphic to the group of $\mathbb{F}_T$-automorphisms of the column space $C(E)$
- $H_E$ is isomorphic to the group of $\mathbb{F}_T$-automorphisms of the row space $R(E)$. 
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$$G_E = \{G : G \text{ is a unit in } M_n(\mathbb{T}) \text{ and } GE = EG\}.$$ 

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**Theorem [IJK]**

Every $\mathbb{FT}$-module automorphism of a projective $n$-polytope

(i) extends to an automorphism of $\mathbb{FT}^n$ and

(ii) is a (classical) affine linear map.
Maximal subgroups of $M_n(FT)$

**Theorem [IJK]**
Let $E$ be an idempotent of rank $n$ in $M_n(FT)$. Then $H_E \cong \mathbb{R} \times \Sigma$, for some $\Sigma \leq S_n$. 

**Corollary [IJK]**
Let $H$ be a maximal subgroup of $M_n(FT)$ containing a rank $k$ idempotent. Then $H \cong \mathbb{R} \times \Sigma$, for some $\Sigma \leq S_k$. 

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Theorem. Then $H \cong \mathbb{R} \times \Sigma$, for some $\Sigma \leq S_k$. 
Let $[n] = \{1, \ldots, n\}$ and let $d : [n] \times [n] \to \mathbb{R}$ be a metric. Consider the $n \times n$ matrix $E$ with $E_{i,j} = -d(i, j)$. Then

- $E$ is symmetric;
- $E \otimes E = E$;
- $C(E)$ has tropical dimension $n$.

**Theorem.** [JK] Let $E$ be an idempotent corresponding to a metric $d$ on $n$ points. Then $H_E \cong \mathbb{R} \times I$, where $I$ is the isometry group of the finite metric space $([n], d)$. 


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**Corollary.** [JK] Let $G$ be a finite group. Then $\mathbb{R} \times G$ is a maximal subgroup of $M_n(\mathbb{F}_T)$, for $n$ sufficiently large.