Green’s $J$-order and the rank of tropical matrices

Marianne Johnson
(joint work with Mark Kambites)
arXiv:1102.2707v1 [math.RA]

Groups and Semigroups: Interactions and Computations,
Lisbon, 29th July 2011
Let \( \mathbb{T} = \mathbb{R} \cup \{-\infty\} \) and define two binary operations on \( \mathbb{T} \) by

\[
\begin{align*}
a \oplus b &:= \max(a, b), \\
a \otimes b &:= a + b,
\end{align*}
\]

\[
(a \oplus -\infty = -\infty \oplus a = a, \quad a \otimes -\infty = -\infty \otimes a = -\infty).
\]

We say that \( \mathbb{T} \) is a (commutative) idempotent semiring.
It is often referred to as the max-plus or tropical semiring.

We also define the finitary tropical semiring,
\( \mathbb{FT} = (\mathbb{R}, \oplus, \otimes) \).

Throughout let \( S = \mathbb{T} \) or \( \mathbb{FT} \).
We study the multiplicative semigroup \( (M_n(S), \otimes) \) of all \( n \times n \) matrices with entries in \( S \).
Tropical convex sets

We write $S^n$ to denote the set of all $n$-tuples $x = (x_1, \ldots, x_n)$ with $x_i \in S$ and extend $\oplus$ to $S^n$ componentwise:

$$(x \oplus y)_i = x_i \oplus y_i.$$  

We also define a scaling action of $S$ on $S^n$:

$$(\lambda \otimes x)_i = \lambda \otimes x_i, \text{ for all } \lambda \in S.$$  

A tropical convex set $X$ in $S^n$ is a subset that is closed under $\oplus$ and scaling.
Green’s relations

Let \( S = \mathbb{T} \) or \( \mathbb{F} \mathbb{T} \) and let \( A, B \in M_n(S) \).

(1) \( A \mathcal{L} B \iff \) row space of \( A \) = row space of \( B \).

(2) \( A \mathcal{R} B \iff \) col. space of \( A \) = col. space of \( B \).

(3) \( A \mathcal{D} B \iff \) row space of \( A \cong \) row space of \( B \\
\iff \) col. space of \( A \cong \) col. space of \( B \\
(Hollings and Kambites, 2010.)

We shall describe Green’s \( \mathcal{J} \)-order and hence the corresponding \( \mathcal{J} \)-relation.
Comparing $\mathcal{D}$ and $\mathcal{J}$

First question should be is $\mathcal{D} = \mathcal{J}$?
Comparing $\mathcal{D}$ and $\mathcal{J}$

First question should be is $\mathcal{D} = \mathcal{J}$?

**Theorem.** (Izhakian and Margolis)

$\mathcal{D} \neq \mathcal{J}$ in $M_n(\mathbb{T})$ for all $n \geq 3$. 
Comparing $\mathcal{D}$ and $\mathcal{J}$

First question should be is $\mathcal{D} = \mathcal{J}$?

**Theorem.** (Izhakian and Margolis) 
$\mathcal{D} \neq \mathcal{J}$ in $M_n(\mathbb{T})$ for all $n \geq 3$.

- We have also constructed a nice example (with pictures!) to show that $\mathcal{D} \neq \mathcal{J}$ for all $n \geq 3$. 
Comparing \( \mathcal{D} \) and \( \mathcal{J} \)

First question should be is \( \mathcal{D} = \mathcal{J} \)?

**Theorem.** (Izhakian and Margolis)
\( \mathcal{D} \neq \mathcal{J} \) in \( M_n(\mathbb{T}) \) for all \( n \geq 3 \).

- We have also constructed a nice example (with pictures!) to show that \( \mathcal{D} \neq \mathcal{J} \) for all \( n \geq 3 \).

**Proposition.** \( \mathcal{D} = \mathcal{J} \) in \( M_2(\mathbb{T}) \).
Comparing $\mathcal{D}$ and $\mathcal{J}$

First question should be is $\mathcal{D} = \mathcal{J}$?

**Theorem.** (Izhakian and Margolis) $\mathcal{D} \neq \mathcal{J}$ in $M_n(\mathbb{T})$ for all $n \geq 3$.

- We have also constructed a nice example (with pictures!) to show that $\mathcal{D} \neq \mathcal{J}$ for all $n \geq 3$.

**Proposition.** $\mathcal{D} = \mathcal{J}$ in $M_2(\mathbb{T})$.

**Theorem.** The $\mathcal{J}$-order in $M_n(\mathbb{FT})$ is inherited from the $\mathcal{J}$-order in $M_n(\mathbb{T})$. 
Comparing $\mathcal{D}$ and $\mathcal{I}$

First question should be is $\mathcal{D} = \mathcal{I}$?

**Theorem.** (Izhakian and Margolis)  
$\mathcal{D} \neq \mathcal{I}$ in $M_n(\mathbb{T})$ for all $n \geq 3$.

- We have also constructed a nice example (with pictures!) to show that $\mathcal{D} \neq \mathcal{I}$ for all $n \geq 3$.

**Proposition.** $\mathcal{D} = \mathcal{I}$ in $M_2(\mathbb{T})$.

**Theorem.** The $\mathcal{I}$-order in $M_n(\mathbb{F}\mathbb{T})$ is inherited from the $\mathcal{I}$-order in $M_n(\mathbb{T})$.

However...

**Theorem.** $\mathcal{D} = \mathcal{I}$ in $M_n(\mathbb{F}\mathbb{T})$ for all $n \geq 1$. 
We define tropical projective space \( \mathcal{P} \mathbb{FT}^n \) by identifying two elements of \( \mathbb{FT}^n \) if one is a finite tropical multiple of the other.
We define **tropical projective space** $\mathcal{P}^{\mathbb{FT}^n}$ by identifying two elements of $\mathbb{FT}^n$ if one is a finite tropical multiple of the other.

Note that we may identify $\mathcal{P}^{\mathbb{FT}^n}$ with $\mathbb{R}^{n-1}$ via

$$[x_1, \ldots, x_n] \mapsto (x_1 - x_n, \ldots, x_{n-1} - x_n).$$
We define **tropical projective space** $\mathcal{P}^{\mathbb{FT}_n}$ by identifying two elements of $\mathbb{FT}_n$ if one is a finite tropical multiple of the other.

Note that we may identify $\mathcal{P}^{\mathbb{FT}_n}$ with $\mathbb{R}^{n-1}$ via

$$[x_1, \ldots, x_n] \mapsto (x_1 - x_n, \ldots, x_{n-1} - x_n).$$

Recall that we have a “distance function” on $\mathbb{FT}_n$ defined by $d_H(x, y) = 0$ if $x$ is a finite tropical multiple of $y$ and

$$d_H(x, y) = \max(y_i - x_i) - \min(y_i - x_i).$$
Tropical projective space

We define tropical projective space $\mathcal{P}^{\mathbb{FT}^n}$ by identifying two elements of $\mathbb{FT}^n$ if one is a finite tropical multiple of the other.

Note that we may identify $\mathcal{P}^{\mathbb{FT}^n}$ with $\mathbb{R}^{n-1}$ via

$$[x_1, \ldots, x_n] \mapsto (x_1 - x_n, \ldots, x_{n-1} - x_n).$$

Recall that we have a “distance function” on $\mathbb{FT}^n$ defined by $d_H(x, y) = 0$ if $x$ is a finite tropical multiple of $y$ and

$$d_H(x, y) = \max(y_i - x_i) - \min(y_i - x_i).$$

It can be shown that $d_H$ is a metric on $\mathcal{P}^{\mathbb{FT}^n}$. 
1. Easy to check that $d_H$ induces the **usual topology** on $\mathbb{R}^{n-1}$.  

The key results used in the proof
1. Easy to check that $d_H$ induces the **usual topology** on $\mathbb{R}^{n-1}$.

Each finitely generated convex set $X \subseteq FT^n$ induces a subset $PX$ of $PFT^n$ termed the **projectivisation** of $X$. 

The key results used in the proof
1. Easy to check that $d_H$ induces the **usual topology** on $\mathbb{R}^{n-1}$.

Each finitely generated convex set $X \subseteq \mathcal{F}\mathcal{T}^n$ induces a subset $\mathcal{P}X$ of $\mathcal{P}\mathcal{F}\mathcal{T}^n$ termed the **projectivisation** of $X$. 
The key results used in the proof

1. Easy to check that $d_H$ induces the **usual topology** on $\mathbb{R}^{n-1}$.

Each finitely generated convex set $X \subseteq FT^n$ induces a subset $\mathcal{P}X$ of $\mathcal{P}FT^n$ termed the **projectivisation** of $X$.

2. The projectivisation of each finitely generated convex set $X \subseteq FT^n$, denoted $\mathcal{P}X$, is a closed and bounded (and hence **compact**) subset of $\mathcal{P}FT^n$. 
The key results used in the proof

1. Easy to check that $d_H$ induces the **usual topology** on $\mathbb{R}^{n-1}$.

Each finitely generated convex set $X \subseteq \mathbb{FT}^n$ induces a subset $P X$ of $P \mathbb{FT}^n$ termed the **projectivisation** of $X$.

2. The projectivisation of each finitely generated convex set $X \subseteq \mathbb{FT}^n$, denoted $P X$, is a closed and bounded (and hence **compact**) subset of $P \mathbb{FT}^n$.

3. **Metric Duality Theorem:**
Let $A \in M_n(\mathbb{FT})$. There exist mutually inverse isometric embeddings between $P R(A)$ and $P C(A)$. 

Comparing $\mathcal{D}$ and $\mathcal{J}$

**Theorem.** $\mathcal{D} = \mathcal{J}$ in $M_n(\mathbb{F} \mathbb{T})$.

**Sketch proof** Clearly $A\mathcal{D}B \Rightarrow A\mathcal{J}B$.
Suppose for contradiction that $A\mathcal{J}B$, but $A\mathcal{D}B$.
Comparing $\mathcal{D}$ and $\mathcal{J}$

**Theorem.** $\mathcal{D} = \mathcal{J}$ in $M_n(\mathbb{FT})$.

**Sketch proof** Clearly $A\mathcal{D}B \Rightarrow A\mathcal{J}B$.
Suppose for contradiction that $A\mathcal{J}B$, but $A\mathcal{D}B$.
Then there is a non-surjective isometric embedding

$$f : \mathcal{P}R(A) \to \mathcal{P}R(A).$$
Comparing $\mathcal{D}$ and $\mathcal{J}$

**Theorem.** $\mathcal{D} = \mathcal{J}$ in $M_n(\mathbb{F} \mathcal{T})$.

**Sketch proof** Clearly $A \mathcal{D} B \Rightarrow A \mathcal{J} B$.
Suppose for contradiction that $A \mathcal{J} B$, but $A \mathcal{D} B$.
Then there is a non-surjective isometric embedding

$$f : \mathcal{P} R(A) \rightarrow \mathcal{P} R(A).$$

Since $f$ is not surjective and has closed image we may choose
$x_0 \in \mathcal{P} R(A)$ and $\varepsilon > 0$ such that $x_0 \notin f(\mathcal{P} R(A))$ and
distance $d_H(x_0, z) \geq \varepsilon$ for all $z \in f(X_0)$. 
Comparing $\mathcal{D}$ and $\mathcal{J}$

**Theorem.** $\mathcal{D} = \mathcal{J}$ in $M_n(\mathbb{FT})$.

**Sketch proof** Clearly $A \mathcal{D} B \Rightarrow A \mathcal{J} B$.

Suppose for contradiction that $A \mathcal{J} B$, but $A \mathcal{D} B$.

Then there is a non-surjective isometric embedding

$$f : \mathcal{P} R(A) \to \mathcal{P} R(A).$$

Since $f$ is not surjective and has closed image we may choose $x_0 \in \mathcal{P} R(A)$ and $\varepsilon > 0$ such that $x_0 \notin f(\mathcal{P} R(A))$ and $d_H(x_0, z) \geq \varepsilon$ for all $z \in f(X_0)$.

Now set $X_i = f^i(\mathcal{P} R(A))$ and let $x_i = f^i(x_0) \in X_i$. Since $f$ is an isometric embedding we have $d_H(x_i, y) \geq \varepsilon$ for all $y \in X_{i+1}$. 
Comparing $\mathcal{D}$ and $\mathcal{J}$

**Theorem.** $\mathcal{D} = \mathcal{J}$ in $M_n(\mathbb{FT})$.

**Sketch proof** Clearly $A \mathcal{D} B \Rightarrow A \mathcal{J} B$.
Suppose for contradiction that $A \mathcal{J} B$, but $A \mathcal{D} B$.
Then there is a non-surjective isometric embedding

$$f : \mathcal{PR}(A) \rightarrow \mathcal{PR}(A).$$

Since $f$ is not surjective and has closed image we may choose $x_0 \in \mathcal{PR}(A)$ and $\varepsilon > 0$ such that $x_0 \notin f(\mathcal{PR}(A))$ and $d_H(x_0, z) \geq \varepsilon$ for all $z \in f(X_0)$.

Now set $X_i = f^i(\mathcal{PR}(A))$ and let $x_i = f^i(x_0) \in X_i$.
Since $f$ is an isometric embedding we have $d_H(x_i, y) \geq \varepsilon$ for all $y \in X_{i+1}$.

In particular $d_h(x_i, x_j) \geq \varepsilon$ for all $j > i$.
This contradicts the compactness of $\mathcal{PR}(A) \subseteq \mathcal{PR}\mathbb{FT}^n = \mathbb{R}^{n-1}$. 
Green’s $\mathcal{J}$-order on $M_n(\mathbb{T})$

**Theorem.** Let $A, B \in M_n(\mathbb{T})$. Then the following are equivalent.

(i) $A \leq_{\mathcal{J}} B$;

(ii) There is a $\mathbb{T}$-linear convex set $X$ such that the row space of $A$ embeds linearly into $X$ and the row space of $B$ surjects linearly onto $X$;

(iii) There is a $\mathbb{T}$-linear convex set $Y$ such that the column space of $A$ embeds linearly into $Y$ and the column space of $B$ surjects linearly onto $Y$. 
**Theorem.** Let $A, B \in M_n(\mathbb{T})$. Then the following are equivalent.

(i) $A \preceq J B$;

(ii) There is a $\mathbb{T}$-linear convex set $X$ such that the row space of $A$ embeds linearly into $X$ and the row space of $B$ surjects linearly onto $X$;

(iii) There is a $\mathbb{T}$-linear convex set $Y$ such that the col. space of $A$ embeds linearly into $Y$ and the col. space of $B$ surjects linearly onto $Y$. 
The rank of a tropical matrix

There are several (non-equivalent) notions of the rank of a tropical matrix. We define three such here...

**factor rank** \( A \) = the minimum \( k \) such that \( A \) can be factored as \( A = CR \) where \( C \) is \( n \times k \) and \( R \) is \( k \times n \)

**det rank** \( A \) = the maximum \( k \) such that \( A \) has a \( k \times k \) minor \( M \) with \( |M|^+ \neq |M|^− \)

**tropical rank** \( A \) = the maximum \( k \) such that \( A \) has a \( k \times k \) minor \( M \) where the max. is achieved uniquely in the permanent of \( M \).
The rank of a tropical matrix

There are several (non-equivalent) notions of the rank of a tropical matrix. We define three such here...

\[
\text{factor rank}(A) = \text{the minimum } k \text{ such that } A \text{ can be factored as } A = CR \text{ where } C \text{ is } n \times k \text{ and } R \text{ is } k \times n
\]

\[
\text{det rank}(A) = \text{the maximum } k \text{ such that } A \text{ has a } k \times k \text{ minor } M \text{ with } |M|^+ \neq |M|^-
\]

\[
\text{tropical rank}(A) = \text{the maximum } k \text{ such that } A \text{ has a } k \times k \text{ minor } M \text{ where the max. is achieved uniquely in the permanent of } M.
\]

**Observation.** The factor rank, det rank, tropical rank (and others) are all \(\mathcal{J}\)-class invariants in \(M_n(\mathbb{T})\).
Comparing $\mathcal{D}$ and $\mathcal{J}$

Example.

$$A = \begin{pmatrix} -\infty & 0 & 1 \\ -\infty & -\infty & 1 \\ 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} -\infty & 0 & 2 \\ -\infty & -\infty & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

- It is easy to see that $C(A) \subseteq C(B)$. Hence $A \leq_{\mathcal{R}} B$.
- It is also easy to see that $R(B) \subseteq R(A)$. Hence $B \leq_{\mathcal{L}} A$.
- Thus we have shown that $A \mathcal{J} B$. 

Claim that $P_{C}(A)$ is not isometric to $P_{C}(B)$. Thus, $A \not\sim_{\mathcal{D}} B$ since any isomorphism between the column spaces would induce an isometry between the projective column spaces.

It follows that $D \neq J$ in $M_{n}(T)$ for all $n \geq 3$. 
Example.

\[ A = \begin{pmatrix} -\infty & 0 & 1 \\ -\infty & -\infty & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -\infty & 0 & 2 \\ -\infty & -\infty & 2 \\ 0 & 0 & 0 \end{pmatrix} \]

- It is easy to see that \( C(A) \subseteq C(B) \). Hence \( A \preceq R B \).
- It is also easy to see that \( R(B) \subseteq R(A) \). Hence \( B \preceq L A \).
- Thus we have shown that \( A \leq J B \).
- Claim that \( PC(A) \) is not isometric to \( PC(B) \). Thus, \( A \not\simeq B \) since any isomorphism between the column spaces would induce an isometry between the projective column spaces.
- It follows that \( D \neq J \) in \( M_n(\mathbb{T}) \) for all \( n \geq 3 \).