Projectivity of Tropical Polytopes

Marianne Johnson
(joint work with Zur Izhakian and Mark Kambites)
arXiv:1106.4525v1 [math.RA]

6th December 2011

Research supported by EPSRC Grant EP/H000801/1, Israel Science Foundation grant number 448/09 and the Mathematisches Forschungsinstitut Oberwolfach.
The tropical semifield

Let $\mathbb{FT} = (\mathbb{R}, \oplus, \otimes)$ where $\oplus$ and $\otimes$ denote two binary operations defined by

$$a \oplus b := \max(a, b), \quad a \otimes b := a + b,$$

$\mathbb{FT}$ is a commutative semigroup; $(\mathbb{FT}, \oplus)$ is a commutative group with identity element $0$; $\otimes$ distributes over $\oplus$; for all $a \in \mathbb{FT}$ we have $a \oplus a = a$. We say that $\mathbb{FT}$ is an idempotent semifield. It is often referred to as the max-plus or tropical semifield.
The tropical semifield

Let $\mathbb{FT} = (\mathbb{R}, \oplus, \otimes)$ where $\oplus$ and $\otimes$ denote two binary operations defined by

$$a \oplus b := \max(a, b), \quad a \otimes b := a + b,$$

- $(\mathbb{FT}, \oplus)$ is a commutative semigroup;
- $(\mathbb{FT}, \otimes)$ is a commutative group with identity element 0;
- $\otimes$ distributes over $\oplus$;
Let $\mathbb{F}_T = (\mathbb{R}, \oplus, \otimes)$ where $\oplus$ and $\otimes$ denote two binary operations defined by

$$a \oplus b := \max(a, b), \quad a \otimes b := a + b,$$

- $(\mathbb{F}_T, \oplus)$ is a commutative semigroup;
- $(\mathbb{F}_T, \otimes)$ is a commutative group with identity element 0;
- $\otimes$ distributes over $\oplus$;
- For all $a \in \mathbb{F}_T$ we have $a \oplus a = a$.

We say that $\mathbb{F}_T$ is an idempotent semifield.
It is often referred to as the max-plus or tropical semifield.
The tropical semifield has applications in areas such as...

- analysis of discrete event systems
- combinatorial optimisation and scheduling problems
- formal languages and automata
- statistical inference
- algebraic geometry
- computing eigenvalues of matrix polynomials...
Motivation

The tropical semifield has applications in areas such as...

- analysis of discrete event systems
- combinatorial optimisation and scheduling problems
- formal languages and automata
- statistical inference
- algebraic geometry
- computing eigenvalues of matrix polynomials...

Problems in application areas typically involve finding solutions to a system of linear equations over the tropical semifield.
Motivation

The tropical semifield has applications in areas such as...

- analysis of discrete event systems
- combinatorial optimisation and scheduling problems
- formal languages and automata
- statistical inference
- algebraic geometry
- computing eigenvalues of matrix polynomials...

Problems in application areas typically involve finding solutions to a system of linear equations over the tropical semifield.

We are therefore interested in properties of matrices with entries in the tropical semifield and their action upon vectors.
Consider the set $M_n(\mathbb{FT})$ of all $n \times n$ matrices over $\mathbb{FT}$. We define multiplication $\otimes$ of tropical matrices as follows:

$$(A \otimes B)_{i,j} = \bigoplus_{k=1}^{n} A_{i,k} \otimes B_{k,j}, \text{ for all } A, B \in M_n(\mathbb{FT}).$$
Tropical matrices

Consider the set $M_n(\mathbb{FT})$ of all $n \times n$ matrices over $\mathbb{FT}$. We define multiplication $\otimes$ of tropical matrices as follows:

$$(A \otimes B)_{i,j} = \bigoplus_{k=1}^{n} A_{i,k} \otimes B_{k,j}, \text{ for all } A, B \in M_n(\mathbb{FT}).$$

Example.

$$\begin{pmatrix} 0 & 1 & 2 \\ 7 & 19 & 3 \\ -5 & 2 & 6 \end{pmatrix} \otimes \begin{pmatrix} -1 & -1 & -2 \\ -20 & 4 & 5 \\ 1 & 2 & 9 \end{pmatrix}$$
Consider the set $M_n(\mathbb{FT})$ of all $n \times n$ matrices over $\mathbb{FT}$. We define multiplication $\otimes$ of tropical matrices as follows:

$$(A \otimes B)_{i,j} = \bigoplus_{k=1}^{n} A_{i,k} \otimes B_{k,j}, \text{ for all } A, B \in M_n(\mathbb{FT}).$$

**Example.**

$$\begin{pmatrix}
0 & 1 & 2 \\
7 & 19 & 3 \\
-5 & 2 & 6
\end{pmatrix} \otimes \begin{pmatrix}
-1 & -1 & -2 \\
-20 & 4 & 5 \\
1 & 2 & 9
\end{pmatrix} = \begin{pmatrix}
3 & 5 & 11 \\
6 & 23 & 24 \\
7 & 8 & 15
\end{pmatrix}$$
Tropical matrices

Consider the set $M_n(\mathbb{F}_T)$ of all $n \times n$ matrices over $\mathbb{F}_T$. We define multiplication $\otimes$ of tropical matrices as follows:

$$(A \otimes B)_{i,j} = \bigoplus_{k=1}^{n} A_{i,k} \otimes B_{k,j}, \text{ for all } A, B \in M_n(\mathbb{F}_T).$$

Example.

$$\begin{pmatrix} 0 & 1 & 2 \\ 7 & 19 & 3 \\ -5 & 2 & 6 \end{pmatrix} \otimes \begin{pmatrix} -1 & -1 & -2 \\ -20 & 4 & 5 \\ 1 & 2 & 9 \end{pmatrix} = \begin{pmatrix} 3 & 5 & 11 \\ 6 & 23 & 24 \\ 7 & 8 & 15 \end{pmatrix}$$

It is easy to see that $(M_n(\mathbb{F}_T), \otimes)$ forms a semigroup.
Tropical vectors

We write $\mathbb{FT}^n$ to denote the set of all $n$-tuples $x = (x_1, \ldots, x_n)$ with $x_i \in \mathbb{FT}$ and extend the addition $\oplus$ to $\mathbb{FT}^n$ componentwise:

$$(x \oplus y)_i = x_i \oplus y_i.$$
We write $\mathbb{FT}^n$ to denote the set of all $n$-tuples $x = (x_1, \ldots, x_n)$ with $x_i \in \mathbb{FT}$ and extend the addition $\oplus$ to $\mathbb{FT}^n$ componentwise:

$$(x \oplus y)_i = x_i \oplus y_i.$$ 

We also define a scaling action of $\mathbb{FT}$ on $\mathbb{FT}^n$:

$$(\lambda \otimes x)_i = \lambda \otimes x_i, \text{ for all } \lambda \in \mathbb{FT}.$$
Tropical vectors

We write $\mathbb{FT}^n$ to denote the set of all $n$-tuples $x = (x_1, \ldots, x_n)$ with $x_i \in \mathbb{FT}$ and extend the **addition** $\oplus$ to $\mathbb{FT}^n$ componentwise:

$$(x \oplus y)_i = x_i \oplus y_i.$$ 

We also define a **scaling** action of $\mathbb{FT}$ on $\mathbb{FT}^n$:

$$(\lambda \otimes x)_i = \lambda \otimes x_i, \text{ for all } \lambda \in \mathbb{FT}.$$ 

Thus $\mathbb{FT}^n$ has the structure of an $\mathbb{FT}$-module.
A **tropical convex set** is a subset $X \subseteq \mathbb{FT}^n$ that is closed under $\oplus$ and scaling (i.e. an $\mathbb{FT}$-submodule of $\mathbb{FT}^n$).
A **tropical convex set** is a subset $X \subseteq \mathbb{FT}^n$ that is closed under $\oplus$ and scaling (i.e. an $\mathbb{FT}$-submodule of $\mathbb{FT}^n$).

If $X$ is finitely generated, we say that $X$ is a **tropical polytope**.
Tropical polytopes

A **tropical convex set** is a subset $X \subseteq \mathbb{FT}^n$ that is closed under $\oplus$ and scaling (i.e. an $\mathbb{FT}$-submodule of $\mathbb{FT}^n$).

If $X$ is finitely generated, we say that $X$ is a **tropical polytope**.

**Example.**
Let $A \in M_n(\mathbb{FT})$. We define the **row space** $R(A) \subseteq \mathbb{FT}^n$ to be the tropical polytope generated by the rows of $A$.

Similarly, we define the **column space** $C(A) \subseteq \mathbb{FT}^n$ to be the tropical polytope generated by the columns of $A$. 

Caution: the row space need not be linearly isomorphic to the column space.
A tropical convex set is a subset $X \subseteq \mathbb{FT}^n$ that is closed under $\oplus$ and scaling (i.e. an $\mathbb{FT}$-submodule of $\mathbb{FT}^n$).

If $X$ is finitely generated, we say that $X$ is a tropical polytope.

**Example.**
Let $A \in M_n(\mathbb{FT})$. We define the row space $R(A) \subseteq \mathbb{FT}^n$ to be the tropical polytope generated by the rows of $A$.

Similarly, we define the column space $C(A) \subseteq \mathbb{FT}^n$ to be the tropical polytope generated by the columns of $A$.

Caution: the row space need not be linearly isomorphic to the column space.
A module $P$ is called **projective** if for every morphism $f : P \to M$ and every surjective morphism $g : N \to M$ there exists a morphism $h : P \to N$ such that $f = g \circ h$. 

We say that $A \in \mathcal{M}_n(\mathbb{F}_T)$ is (von Neumann) **regular** if there exists $B \in \mathcal{M}_n(\mathbb{F}_T)$ such that $A \otimes B \otimes A = A$.

**Theorem 1.** $A$ is regular $\iff$ $R(A)$ is projective $\iff$ $C(A)$ is projective.

**Sketch proof:**

$\implies$ $A$ is regular $\iff$ it is "D-related" to an idempotent.

$\implies$ Hollings and Kambites, 2010: Tropical matrices are "D-related" $\iff$ their row spaces (dually, column spaces) are isomorphic.

$\implies$ A tropical polytope $X \subseteq \mathcal{F}_T^n$ is projective if and only if it is isomorphic to the image of an idempotent.
A module $P$ is called **projective** if for every morphism $f : P \to M$ and every surjective morphism $g : N \to M$ there exists a morphism $h : P \to N$ such that $f = g \circ h$.

We say that $A \in M_n(\mathbb{FT})$ is (von Neumann) **regular** if there exists $B \in M_n(\mathbb{FT})$ such that $A \otimes B \otimes A = A$. 

**Theorem 1.** $A$ is regular $\iff R(A)$ is projective $\iff C(A)$ is projective.

**Sketch proof:**
1. $A$ is regular $\iff$ it is "$D$-related" to an idempotent.
2. Hollings and Kambites, 2010: Tropical matrices are "$D$-related" $\iff$ their row spaces (dually, column spaces) are isomorphic.
3. A tropical polytope $X \subseteq \mathbb{FT}^n$ is projective if and only if it is isomorphic to the image of an idempotent.
Projectivity and regularity

A module $P$ is called **projective** if for every morphism $f : P \to M$ and every surjective morphism $g : N \to M$ there exists a morphism $h : P \to N$ such that $f = g \circ h$.

We say that $A \in M_n(\mathbb{F}T)$ is (von Neumann) **regular** if there exists $B \in M_n(\mathbb{F}T)$ such that $A \otimes B \otimes A = A$.

**Theorem 1.**

$A$ is regular $\iff R(A)$ is projective $\iff C(A)$ is projective.
A module $P$ is called **projective** if for every morphism $f : P \rightarrow M$ and every surjective morphism $g : N \rightarrow M$ there exists a morphism $h : P \rightarrow N$ such that $f = g \circ h$.

We say that $A \in M_n(\mathbb{FT})$ is (von Neumann) **regular** if there exists $B \in M_n(\mathbb{FT})$ such that $A \otimes B \otimes A = A$.

**Theorem 1.**

$A$ is regular $\iff R(A)$ is projective $\iff C(A)$ is projective.

**Sketch proof:**

- $A$ is regular $\iff$ it is “$\mathcal{D}$-related” to an idempotent.
Projectivity and regularity

A module $P$ is called **projective** if for every morphism $f : P \to M$ and every surjective morphism $g : N \to M$ there exists a morphism $h : P \to N$ such that $f = g \circ h$.

We say that $A \in M_n(\mathbb{FT})$ is (von Neumann) **regular** if there exists $B \in M_n(\mathbb{FT})$ such that $A \otimes B \otimes A = A$.

**Theorem 1.**

$A$ is regular $\iff R(A)$ is projective $\iff C(A)$ is projective.

**Sketch proof:**

- $A$ is regular $\iff$ it is “$D$-related” to an idempotent.
- Hollings and Kambites, 2010:
  Tropical matrices are “$D$-related” $\iff$ their row spaces (dually, column spaces) are isomorphic.
A module $P$ is called **projective** if for every morphism $f : P \to M$ and every surjective morphism $g : N \to M$ there exists a morphism $h : P \to N$ such that $f = g \circ h$.

We say that $A \in M_n(\mathbb{FT})$ is (von Neumann) **regular** if there exists $B \in M_n(\mathbb{FT})$ such that $A \otimes B \otimes A = A$.

**Theorem 1.**

$A$ is regular $\iff R(A)$ is projective $\iff C(A)$ is projective.

**Sketch proof:**

- $A$ is regular $\iff$ it is “$\mathcal{D}$-related” to an idempotent.
- Hollings and Kambites, 2010:
  Tropical matrices are “$\mathcal{D}$-related” $\iff$ their row spaces (dually, column spaces) are isomorphic.
- A tropical polytope $X \subseteq \mathbb{FT}^n$ is projective if and only if it is isomorphic to the image of an idempotent.
Projectivity and regularity

Recall that \( A \in M_n(\mathbb{F}_T) \) is regular if there exists \( B \in M_n(\mathbb{F}_T) \) such that \( A \otimes B \otimes A = A \).

Example.

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 2 & 0 \\
2 & 2 & 0
\end{pmatrix} \otimes \begin{pmatrix}
-2 & -2 & -2 \\
-2 & -2 & -4 \\
0 & -2 & -2
\end{pmatrix} \otimes \begin{pmatrix}
0 & 0 & 0 \\
0 & 2 & 0 \\
2 & 2 & 0
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 2 & 0 \\
2 & 2 & 0
\end{pmatrix}
\]
Projectivity and regularity

Recall that $A \in M_n(\mathbb{F}_T)$ is **regular** if there exists $B \in M_n(\mathbb{F}_T)$ such that $A \otimes B \otimes A = A$.

**Example.**

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 2 & 0 \\
2 & 2 & 0 \\
\end{pmatrix} \otimes \begin{pmatrix}
-2 & -2 & -2 \\
-2 & -2 & -4 \\
0 & -2 & -2 \\
\end{pmatrix} \otimes \begin{pmatrix}
0 & 0 & 0 \\
0 & 2 & 0 \\
2 & 2 & 0 \\
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 2 & 0 \\
2 & 2 & 0 \\
\end{pmatrix}
\]

Can we give a **geometric** characterisation of the projective tropical polytopes (and hence, the regular matrices in $M_n(\mathbb{F}_T)$)?
Let \( X \subseteq \mathbb{FT}^n \) be a tropical polytope.
Dimensions of tropical polytopes

Let $X \subseteq \mathbb{FT}^n$ be a tropical polytope.

The **tropical dimension** of $X$ is the maximum topological dimension of $X$ regarded as a subset of $\mathbb{R}^n$. 
Dimensions of tropical polytopes

Let $X \subseteq \mathbb{FT}^n$ be a tropical polytope.

The **tropical dimension** of $X$ is the maximum topological dimension of $X$ regarded as a subset of $\mathbb{R}^n$.

We say that $X$ has **pure tropical dimension** $k$ if every open subset of $X$ has topological dimension $k$. 
Let $X \subseteq \mathbb{FT}^n$ be a tropical polytope.

The **tropical dimension** of $X$ is the maximum topological dimension of $X$ regarded as a subset of $\mathbb{R}^n$.

We say that $X$ has **pure tropical dimension** $k$ if every open subset of $X$ has topological dimension $k$.

The **generator dimension** of $X$ is the minimum cardinality of a generating set for $X$. 

Lemma. The dual dimension of $X$ is equal to the generator dimension of $\mathcal{C}(A)$, where $A$ is any matrix satisfying $X = \mathbb{R}(A)$. 

Let $X \subseteq \mathbb{FT}^n$ be a tropical polytope.

The **tropical dimension** of $X$ is the maximum topological dimension of $X$ regarded as a subset of $\mathbb{R}^n$.

We say that $X$ has **pure tropical dimension** $k$ if every open subset of $X$ has topological dimension $k$.

The **generator dimension** of $X$ is the minimum cardinality of a generating set for $X$.

The **dual dimension** of $X$ is the minimum $k$ such that $X$ embeds linearly into $\mathbb{FT}^k$. 

Lemma. The dual dimension of $X$ is equal to the generator dimension of $C(A)$, where $A$ is any matrix satisfying $X = R(A)$. 


Let $X \subseteq \mathbb{F}_T^n$ be a tropical polytope.

The **tropical dimension** of $X$ is the maximum topological dimension of $X$ regarded as a subset of $\mathbb{R}^n$.

We say that $X$ has **pure tropical dimension** $k$ if every open subset of $X$ has topological dimension $k$.

The **generator dimension** of $X$ is the minimum cardinality of a generating set for $X$.

The **dual dimension** of $X$ is the minimum $k$ such that $X$ embeds linearly into $\mathbb{F}_T^k$.

**Lemma.** The dual dimension of $X$ is equal to the generator dimension of $C(A)$, where $A$ is any matrix satisfying $X = R(A)$. 
Theorem 2. Let $X \subseteq \mathbb{FT}^n$ be a tropical polytope. Then
$X$ is projective $\iff X$ has pure tropical dimension and
trop. dim $=$ gen. dim $=$ dual dim.
Theorem 2. Let $X \subseteq \mathbb{FT}^n$ be a tropical polytope. Then $X$ is projective $\iff$ $X$ has pure tropical dimension and $\text{trop. dim} = \text{gen. dim} = \text{dual dim}$.

Sketch proof:
We make use of the fact that a tropical polytope $X \subseteq \mathbb{FT}^n$ is projective $\iff$ it is isomorphic to the image of an idempotent.
Theorem 2. Let $X \subseteq \mathbb{FT}^n$ be a tropical polytope. Then $X$ is projective $\iff X$ has pure tropical dimension and $\text{trop. dim} = \text{gen. dim} = \text{dual dim}$.

Sketch proof:
We make use of the fact that a tropical polytope $X \subseteq \mathbb{FT}^n$ is projective $\iff$ it is isomorphic to the image of an idempotent.

$\Rightarrow$: Enough to show that the column space of any idempotent matrix has pure tropical dimension $= \text{generator dimension} = \text{dual dimension}$. 

$\Leftarrow$: If $X$ has dual dimension is $k$ then, by definition, $X \sim Y$ for some $Y \subseteq \mathbb{FT}^k$ with all dimensions equal to $k$. Enough to show that every such maximal-dimension tropical polytope in $\mathbb{FT}^k$ is isomorphic to the image of an idempotent.
**Theorem 2.** Let \( X \subseteq \mathbb{T}^n \) be a tropical polytope. Then
\( X \) is projective \( \iff \) \( X \) has pure tropical dimension and
trop. dim = gen. dim = dual dim.

**Sketch proof:**
We make use of the fact that a tropical polytope \( X \subseteq \mathbb{T}^n \) is
projective \( \iff \) it is isomorphic to the image of an idempotent.

\( \Rightarrow \): Enough to show that the column space of any idempotent
matrix has pure tropical dimension = generator dimension =
dual dimension.

\( \Leftarrow \): If \( X \) has dual dimension is \( k \) then, by definition, \( X \cong Y \) for
some \( Y \subseteq \mathbb{T}^k \) with all dimensions equal to \( k \).
Theorem 2. Let $X \subseteq \mathbb{FT}^n$ be a tropical polytope. Then $X$ is projective $\iff$ $X$ has pure tropical dimension and $\text{trop. dim} = \text{gen. dim} = \text{dual dim}$.

Sketch proof:
We make use of the fact that a tropical polytope $X \subseteq \mathbb{FT}^n$ is projective $\iff$ it is isomorphic to the image of an idempotent.

$\Rightarrow$: Enough to show that the column space of any idempotent matrix has pure tropical dimension $= \text{generator dimension} = \text{dual dimension}$.

$\Leftarrow$: If $X$ has dual dimension is $k$ then, by definition, $X \cong Y$ for some $Y \subseteq \mathbb{FT}^k$ with all dimensions equal to $k$.

Enough to show that every such maximal-dimension tropical polytope in $\mathbb{FT}^k$ is isomorphic to the image of an idempotent.
Corollary. Let $A \in M_n(\mathbb{F}_T)$. Then

$A$ is regular $\iff R(A)$ and $C(A)$ have the same pure tropical dimension equal to their generator dimension.
Corollary. Let $A \in M_n(\mathbb{FT})$. Then

$A$ is regular $\iff R(A)$ and $C(A)$ have the same pure tropical dimension equal to their generator dimension.

Proof:

- Theorem 1: $A$ is regular $\iff R(A)$ and $C(A)$ are projective.
**Corollary.** Let $A \in M_n(\mathbb{FT})$. Then

$A$ is regular $\iff R(A)$ and $C(A)$ have the same pure tropical dimension equal to their generator dimension.

**Proof:**

- Theorem 1: $A$ is regular $\iff R(A)$ and $C(A)$ are projective.

- Theorem 2: $R(A)$ projective $\iff R(A)$ has pure tropical dim. and tropical dim. $=$ generator dim. $=$ dual dim.
**Corollary.** Let $A \in M_n(\mathbb{FT})$. Then

$A$ is regular $\iff R(A)$ and $C(A)$ have the same pure tropical dimension equal to their generator dimension.

**Proof:**

- **Theorem 1:** $A$ is regular $\iff R(A)$ and $C(A)$ are projective.
- **Theorem 2:** $R(A)$ projective $\iff R(A)$ has pure tropical dim. and tropical dim. $=$ generator dim. $=$ dual dim.
- **Lemma:** The dual dimension of $R(A)$ is equal to the generator dimension of $C(A)$. 
The rank of a tropical matrix

There are several (non-equivalent) notions of the rank of a tropical matrix:

- **tropical rank** \( \text{tropical rank}(A) \) = tropical dimension of its row or col. space.
- **row rank** \( \text{row rank}(A) \) = generator dimension of row space of \( A \).
- **column rank** \( \text{column rank}(A) \) = generator dimension of col. space of \( A \).
- **factor rank** \( \text{factor rank}(A) \) = the minimum \( k \) such that \( A \) can be factored as \( A = CR \) where \( C \) is \( n \times k \) and \( R \) is \( k \times n \).
- **det rank** \( \text{det rank}(A) \) = the maximum \( k \) such that \( A \) has a \( k \times k \) minor \( M \) with \( |M|^+ \neq |M|^− \)
The rank of a tropical matrix

There are several (non-equivalent) notions of the rank of a tropical matrix:

- **tropical rank** \( A \) = tropical dimension of its row or col. space.
- **row rank** \( A \) = generator dimension of row space of \( A \).
- **column rank** \( A \) = generator dimension of col. space of \( A \).
- **factor rank** \( A \) = the minimum \( k \) such that \( A \) can be factored as \( A = CR \) where \( C \) is \( n \times k \) and \( R \) is \( k \times n \).
- **det rank** \( A \) = the maximum \( k \) such that \( A \) has a \( k \times k \) minor \( M \) with \( |M|^+ \neq |M|^− \).

Also, Kapranov rank, Gondran-Minoux row rank, Gondran-Minoux column rank etc.
The rank of a tropical matrix

There are several (non-equivalent) notions of the rank of a tropical matrix:

- **tropical rank** \( \text{tropical rank}(A) \) = tropical dimension of its row or col. space.
- **row rank** \( \text{row rank}(A) \) = generator dimension of row space of \( A \).
- **column rank** \( \text{column rank}(A) \) = generator dimension of col. space of \( A \).
- **factor rank** \( \text{factor rank}(A) \) = the minimum \( k \) such that \( A \) can be factored as \( A = CR \) where \( C \) is \( n \times k \) and \( R \) is \( k \times n \).
- **det rank** \( \text{det rank}(A) \) = the maximum \( k \) such that \( A \) has a \( k \times k \) minor \( M \) with \( |M|^+ \neq |M|^− \).

Also, Kapranov rank, Gondran-Minoux row rank, Gondran-Minoux column rank etc.

**Corollary.** Let \( A \in M_n(\mathbb{FT}) \) be a regular matrix. Then all these notions of rank coincide.